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— **Electromagnetic Theory: Steady Current and Magnetostatics**

1. 9.15

The Resistance of an Ohmic Sphere

A current I flows up the z-axis and is intercepted by an origin-centered sphere with radius R and conductivity σ. The current enters and exits the sphere through small conducting electrodes which occupy the portion of the sphere's surface defined by $\theta \le \alpha$ and $\pi - \alpha \le \theta \le \pi$. Derive an expression for the resistance of the sphere to the flowing current. Assume that $\alpha \ll 1$ and comment on the limit $\alpha \to 0$. Hint:

$$(2\ell + 1)\int_{x_1}^{x_2} dx\, P_\ell(x) = [P_{\ell+1}(x) - P_{\ell-1}(x)]_{x_1}^{x_2}.$$

1.1. Solution

This problem boils down to finding the potential difference at the terminals in order to construct

$R = \frac{V}{I} = \frac{\phi_\pi - \phi_0}{I}$

To begin the problem, I search for the general form of the potential due to a steady current, then attempt to evaluate at both terminals:

$\Phi(\mathbf{r}, \theta) = \sum_l A_l r^l P_l(\cos\theta)$
and knowing that for a spherical conductor, the electric field is zero everywhere, besides in this situation where we have electrodes. Clearly, we have nonzero E at the terminals:

$E|_{r=R} = -\frac{\partial \Phi}{\partial r}|_{r=R} = -\frac{\partial}{\partial r}\sum_l A_l r^l P_l(\cos\theta) = -\sum_l l A_l R^{l-1} P_l(\cos\theta)$

Where we then use the classic trick in these problems. Multiply through by $\sin\theta$ and $P_m(\cos\theta)$ and integrating through $\cos\theta$

$\int_{-1}^{1} E(r,\theta) P_m(\cos\theta) d(\cos\theta) = -\int_{-1}^{1}\sum_l l A_l R^{l-1} P_l(\cos\theta) P_m(\cos\theta) d(\cos\theta)$
$= -\sum_l l A_l R^{l-1}\int_{-1}^{1} P_l(\cos\theta) P_m(\cos\theta) d(\cos\theta) = -\sum_l l A_l R^{l-1}\left[\frac{2}{2m+1}\delta_{ml}\right] = -\frac{2m}{2m+1} A_m R^{m-1}$
and

$\int_{-1}^{1} E(r,\cos\theta) P_m(\cos\theta) d(\cos\theta) = E(r,\cos\theta)\frac{1}{2m+1}\left[P_{m+1}(x) - P_{m-1}(x)\right] = \frac{E}{2m+1}[P_{m+1}(\cos\alpha) - P_{m-1}(\cos\alpha) - P_{m+1}(-\cos\alpha) + P_{m-1}(-\cos\alpha)] = \frac{2E}{2m+1}[P_{m-1}(-\cos\alpha) - P_{m+1}(\cos\alpha)]$

Therefore, we can find A_m:

$A_m = -\frac{2m+1}{2mR^{m-1}}\frac{2E}{2m+1}[P_{m-1}(-\cos\alpha) - P_{m+1}(\cos\alpha)] = \frac{E}{mR^{m-1}}[P_{m+1}(\cos\alpha) - P_{m-1}(-\cos\alpha)]$

From this, we can reconstruct the potential difference:

$V = \Phi(R,\pi) - \Phi(R,0) = \sum_l A_l r^l [P_l(\cos\pi) - P_l(\cos 0)] = \sum_l \frac{Er^l}{lR^{l-1}}[P_{l+1}(\cos\alpha) - P_{l-1}(-\cos\alpha)][P_l(\cos\pi) - P_l(\cos 0)] = \sum_l \frac{Er^l}{lR^{l-1}}[P_{l+1}(\cos\alpha) - P_{l-1}(-\cos\alpha)][P_l(-1) - P_l(1)]$

All together,

$$V = \Phi(R,\pi) - \Phi(R,0) = \boxed{\sum_l \frac{2Er^l}{lR^{l-1}}[P_{l-1}(-\cos\alpha) - P_{l+1}(\cos\alpha)]}$$

And finding R from this,

$$\bar{R} = \tfrac{V}{I} = \tfrac{\Phi(R,\pi)-\Phi(R,0)}{I} = \boxed{\frac{2E}{IR}\sum_l \frac{1}{l}\left(\frac{r}{R}\right)^l [P_{l-1}(-\cos\alpha) - P_{l+1}(\cos\alpha)]}$$

2. 10.15

A Spinning Spherical Shell of Charge

A charge Q is uniformly distributed over the surface of a sphere of radius R. The sphere spins at a constant angular frequency with $\omega = \dot{\theta}$. Use $\mathbf{B} = -\nabla\psi$ to find the magnetic field everywhere. Hint: See Appendix C.1.1.

2.1. Solution

The problem boils down to employing a similar tactic to the spherical multipole expansion for the electrostatic potential, except now we are working with magnetic fields:

$\psi(\mathbf{r},\theta) = \sum_l A_l r^l P_l(\cos\theta)$ with $r < r'$

$\psi(\mathbf{r},\theta) = \sum_l \frac{B_l}{r^{l+1}} P_l(\cos\theta)$ with $r > r'$

Knowing $\mathbf{K} = \sigma\mathbf{v} = \frac{Q}{4\pi R^2}\mathbf{v}$:

And using the differential velocity element:

$\frac{d\mathbf{r}}{dt} = \cancel{\dot{r}\hat{r}} + \cancel{r\dot{\theta}\hat{\theta}} + r\dot{\phi}\sin\theta\hat{\phi} = r\dot{\phi}\sin\theta\hat{\phi} = R\omega\sin\theta\hat{\phi}$

$\mathbf{K} = \frac{Q}{4\pi R^2}R\omega\sin\theta\hat{\phi} = \frac{Q\omega}{4\pi R}\sin\theta\hat{\phi}$

and employing boundary conditions to solve for the coefficients:

$$\mathbf{n_2}\cdot(\mathbf{B_{out}} - \mathbf{B_{in}}) = 0 \rightarrow \mathbf{n_2}\cdot\left(\frac{\partial\psi_{\mathbf{out}}}{\partial r} - \frac{\partial\psi_{\mathbf{in}}}{\partial r}\right) = 0$$

$$\mathbf{n_2}\times(\mathbf{B_{out}} - \mathbf{B_{in}}) = \mu_0\mathbf{K}(\theta) \rightarrow \mathbf{n_2}\times\left(\frac{\partial\psi_{\mathbf{out}}}{\partial\theta} - \frac{\partial\psi_{\mathbf{in}}}{\partial\theta}\right) = \mu_0\mathbf{K}(\theta)$$

Using the first,

$\frac{\partial}{\partial r}\left[\sum_l A_l R^l P_l(\cos\theta) - \sum_l \frac{B_l}{R^{l+1}}P_l(\cos\theta)\right] = 0$

$\cancel{\sum_l} lA_lR^{l-1}\cancel{P_l(\cos\theta)} = \cancel{\sum_l} -(l+1)\frac{B_l}{R^{l+2}}\cancel{P_l(\cos\theta)} \rightarrow lA_lR^{l-1} = -(l+1)\frac{B_l}{R^{l+2}} \rightarrow \boxed{A_l = \frac{-(l+1)}{l}\frac{B_l}{R^{2l+1}}}$

Using the second,

$\frac{1}{R}\frac{\partial}{\partial\theta}\left[\sum_l A_l R^l P_l(\cos\theta) - \sum_l \frac{B_l}{R^{l+1}} P_l(\cos\theta)\right] = \mu_0 \mathbf{K}(\theta)$

$\sum_l A_l R^{l-1}\frac{\partial P_l(\cos\theta)}{\partial\theta} - \sum_l \frac{B_l}{R^{l+2}}\frac{\partial P_l(\cos\theta)}{\partial\theta} = \mu_0\mathbf{K}(\theta) = \mu_0\frac{Q\omega}{4\pi R}\sin\theta\hat{\phi}$

Where we can imply $l=1$ from knowing the form of the legendre polynomials and $\mathbf{K}(\theta)$:

$A_1\cancel{(-\sin\theta)} - \frac{B_1}{R^3}\cancel{(-\sin\theta)} = \mu_0\frac{Q\omega}{4\pi R}\cancel{\sin\theta}$

$-A_1 + \frac{B_1}{R^3} = \mu_0\frac{Q\omega}{4\pi R} \to \frac{(1+1)}{1}\frac{B_l}{R^3} + \frac{B_1}{R^3} = \mu_0\frac{Q\omega}{4\pi R}$

$B_1\left(\frac{3}{R^2}\right) = \mu_0\frac{Q\omega}{4\pi} \to B_1 = \mu_0\frac{QR^2\omega}{12\pi}$

$A_1 = -2\frac{B_1}{R^3} = \mu_0\frac{QR^2\omega}{12\pi}\frac{-2}{R^3} = -\mu_0\frac{Q\omega}{6\pi R}$

$\psi_{in}(r,\theta) = -\mu_0\frac{Q\omega}{6\pi R}rP_1(\cos\theta) = \boxed{-\mu_0\frac{Q\omega}{6\pi}\frac{r}{R}\cos\theta}$

$\psi_{out}(r,\theta) = \frac{B_1}{r^2}P_1(\cos\theta) = \boxed{\mu_0\frac{Q\omega}{12\pi}\left(\frac{R}{r}\right)^2\cos\theta}$

$\mathbf{B_{in}} = -\nabla\psi_{in} = \boxed{\mu_0\frac{Q\omega}{6\pi}\frac{\cos\theta}{R}\hat{r} - \mu_0\frac{Q\omega}{6\pi}\frac{\sin\theta}{R}\hat{\theta}}$

$\mathbf{B_{out}} = -\nabla\psi_{out} = \boxed{\mu_0\frac{Q\omega}{6\pi}\frac{R^2}{r^3}\cos\theta\hat{r} + \mu_0\frac{Q\omega}{12\pi}\frac{R^2}{r^3}\sin\theta\hat{\theta}}$

3. 10.21

Consequences of Gauge Choices

(a) Show by direct calculation that the Coulomb gauge condition $\nabla\cdot\mathbf{A} = 0$ applies to

$$\mathbf{A}(\mathbf{r}) = \frac{\mu_0}{4\pi}\int d^3\mathbf{r}'\,\frac{\mathbf{j}(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|}.$$

(b) Find the choice of gauge where a valid representation of the vector potential is

$$\mathbf{A}(\mathbf{r}) = \frac{1}{4\pi}\int d^3\mathbf{r}'\,\mathbf{B}(\mathbf{r}')\times\frac{(\mathbf{r}-\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|^3}.$$

3.1. part (a):

By the definition of the Coulomb gauge:

$\nabla\cdot\mathbf{A}(\mathbf{r}) = \nabla\cdot\left[\frac{\mu_0}{4\pi}\int d^3\mathbf{r}'\frac{\mathbf{j}(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|}\right] = \frac{\mu_0}{4\pi}\int d^3\mathbf{r}'\mathbf{j}(\mathbf{r}')\nabla\cdot\left[\frac{1}{|\mathbf{r}-\mathbf{r}'|}\right] = -\frac{\mu_0}{4\pi}\int d^3\mathbf{r}'\mathbf{j}(\mathbf{r}')\nabla'\cdot\left[\frac{1}{|\mathbf{r}-\mathbf{r}'|}\right]$

Then using the identity:

$\nabla(ab) = a\nabla b + b\nabla a$

And solving for $a\nabla b$, then using it on the integrand from above:

$a\nabla b = \nabla(ab) - b\nabla a$

$\nabla \cdot \mathbf{A}(\mathbf{r}) = -\frac{\mu_0}{4\pi}\left[\int \nabla' \cdot \frac{\mathbf{j}(\mathbf{r'})}{|\mathbf{r}-\mathbf{r'}|} d^3\mathbf{r'} - \int \frac{\mathbf{1}}{|\mathbf{r}-\mathbf{r'}|}\nabla' \cdot j(\mathbf{r'}) d^3\mathbf{r'}\right]$

And by the statement of magnetostatics, $\nabla' \cdot j(\mathbf{r'}) = 0$

$\nabla \cdot \mathbf{A}(\mathbf{r}) = -\frac{\mu_0}{4\pi}\left[\int \nabla' \cdot \frac{\mathbf{j}(\mathbf{r'})}{|\mathbf{r}-\mathbf{r'}|} d^3\mathbf{r'} - \cancel{\int \frac{\mathbf{1}}{|\mathbf{r}-\mathbf{r'}|}\nabla' \cdot j(\mathbf{r'}) d^3\mathbf{r'}}\right] = -\frac{\mu_0}{4\pi}\int \nabla' \cdot \frac{\mathbf{j}(\mathbf{r'})}{|\mathbf{r}-\mathbf{r'}|} d^3\mathbf{r'} = -\frac{\mu_0}{4\pi}\oint_S \left(\frac{\mathbf{j}(\mathbf{r'})}{|\mathbf{r}-\mathbf{r'}|}\right) \cdot d\mathbf{S'}$

And by locality of $\mathbf{j}(\mathbf{r'})$ we can choose a surface area such that the integral vanishes. Hence,

$\nabla \cdot \mathbf{A}(\mathbf{r}) = \cancel{-\frac{\mu_0}{4\pi}\oint_S \left(\frac{\mathbf{j}(\mathbf{r'})}{|\mathbf{r}-\mathbf{r'}|}\right) \cdot d\mathbf{S'}} = \boxed{0}$

3.2. part (b):

I make an argument here by constructing $\mathbf{B}$ first, then concluding a fact about the result.

$\mathbf{B}(\mathbf{r}) = \nabla\times\mathbf{A}(\mathbf{r}) = \frac{1}{4\pi}\nabla\times\left(\int d^3\mathbf{r'}\,\mathbf{B}(\mathbf{r'}) \times \frac{(\mathbf{r}-\mathbf{r'})}{|\mathbf{r}-\mathbf{r'}|^3}\right) = \frac{1}{4\pi}\int d^3\mathbf{r'}(\nabla \times \mathbf{B}(\mathbf{r'}))\times\frac{(\mathbf{r}-\mathbf{r'})}{|\mathbf{r}-\mathbf{r'}|^3} = \frac{\mu_0}{4\pi}\left[\int d^3\mathbf{r'}\mathbf{j}(\mathbf{r'}) \times \frac{(\mathbf{r}-\mathbf{r'})}{|\mathbf{r}-\mathbf{r'}|^3}\right]$

Which only holds true for the coulomb gauge, since this is the Biot-Savart result.

$\boxed{\nabla \cdot \mathbf{A}(\mathbf{r}) = 0}$ for this given $\mathbf{A}(\mathbf{r})$

4. 10.23

A Geometry of Aharonov and Bohm

(a) Find the vector potential inside and outside a solenoid that generates a magnetic field $\mathbf{B} = B\hat{z}$ inside an infinite cylinder of radius R. Work in the Coulomb gauge.

(b) The Aharonov-Bohm effect occurs because the magnetic flux $\Phi_B = \int \mathbf{B} \cdot d\mathbf{s}$ is non-zero when the integration circuit is, say, the rim of a disk of radius $\rho > R$ which lies perpendicular to the solenoid axis. Show that $\mathbf{A'} = \mathbf{A} + \nabla\chi$ with $\chi = -\phi\Phi_B/(2\pi)$ (where ϕ is the angle in cylindrical coordinates) leads to identically zero vector potential outside the solenoid.

(c) The result in part (b) implies that we could eliminate the Aharonov-Bohm effect by a gauge transformation. Show, however, that the new magnetic field corresponds to a different physical problem, where

$$\mathbf{B'} = \mathbf{B} - \hat{\mathbf{z}}\frac{\Phi}{2\pi\rho}\delta(\rho).$$

5. Solution

5.1. part (a)

Since we know,

$\nabla \cdot \mathbf{A} = \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z} = 0$ and $\mathbf{B} = \nabla \times \mathbf{A} = B\hat{\mathbf{z}}$

This implies that $\mathbf{A}$ must be $\mathbf{A} = A_x\hat{\mathbf{x}} + A_y\hat{\mathbf{y}} + 0\hat{\mathbf{z}}$

I will use this argument in a bit to put a direction to A. Using Zangwill 10.68:

$\Phi_B = \int_C d\mathbf{l} \cdot \mathbf{A} \rightarrow |\mathbf{B}|(\pi r^2) = \int_C \mathbf{A} \cdot d\mathbf{l}$

$B\pi r^2 = |\mathbf{A}|(2\pi r)$ where we can then just proceed with a Gauss' law-type calculation.

inside: $r < R$: $\boxed{|\mathbf{A}| = \dfrac{Br}{2} = \dfrac{Br}{2}(-\sin\phi\hat{\mathbf{x}} + \cos\phi\hat{\mathbf{y}})} = \frac{Br}{2}\hat{\phi}$

outside: $r > R$: $\boxed{|\mathbf{A}| = \dfrac{BR^2}{2r} = \dfrac{BR^2}{2r}(-\sin\phi\hat{\mathbf{x}} + \cos\phi\hat{\mathbf{y}})} = \frac{BR^2}{2r}\hat{\phi}$

Which ensures no magnitude change of $\mathbf{A}$, but also promises the correct direction as promised earlier in this part.

5.2. part (b)

We directly calculate $\nabla\chi$ and add it to our previous result:

$\nabla\chi = \frac{1}{r}\left(\frac{\partial\chi}{\partial\phi}\hat{\phi}\right) = \frac{1}{r}\left(\frac{-\Phi_B}{2\pi}\phi\hat{\phi}\right) = \frac{-\Phi_B}{2\pi r}(1)\hat{\phi} = \boxed{\dfrac{-\Phi_B}{2\pi r}\hat{\phi}}$

$\mathbf{A}'_{\mathbf{out}} = \mathbf{A}_{\mathbf{out}} + \nabla\chi = \frac{BR^2}{2r}\hat{\phi} + \frac{-\Phi_B}{2\pi r}\hat{\phi}$

But,

$\Phi_B = \int \mathbf{A}\cdot d\mathbf{l} = \int\left(\frac{BR^2}{2r}\hat{\phi}\right)\cdot\left(rd\phi\hat{\phi}\right) = B\pi R^2$

So then it becomes clear that outside the solenoid,

$\mathbf{A}'_{\mathbf{out}} = \frac{BR^2}{2r}\hat{\phi} + \frac{-\Phi_B}{2\pi r}\hat{\phi} = \frac{BR^2}{2r}\hat{\phi} - \frac{B\pi R^2}{2\pi r}\hat{\phi} = \boxed{0}$

5.3. part (c)

$\mathbf{B} = \nabla\times\mathbf{A} = \nabla\times\left(\frac{BR^2}{2r}\hat{\phi}\right) = \frac{BR^2}{2}\nabla\times\left(\frac{1}{r}\hat{\phi}\right) = \frac{BR^2}{2}\left(\frac{\delta(r)}{r}\hat{\mathbf{z}}\right)$

But we know from part (b), that

$\Phi_B = B\pi R^2 \to BR^2 = \Phi_B/\pi$

Therefore, $\mathbf{B}(\mathbf{r}) = \frac{\Phi}{2\pi}\left(\frac{\delta(r)}{r}\hat{\mathbf{z}}\right)$

And since we know that $\mathbf{B}' = \nabla\times\mathbf{A}' = 0$, we can conclude that :

$\mathbf{B}' = \mathbf{B} + f(r) = 0$, where I am calling $f(r)$ some function that ensures that $\mathbf{B}' = 0$.

$\mathbf{B}' = \frac{\Phi}{2\pi}\left(\frac{\delta(r)}{r}\hat{\mathbf{z}}\right) + f(r) = 0$ implies that $f(r) = -\frac{\Phi}{2\pi}\left(\frac{\delta(r)}{r}\hat{\mathbf{z}}\right)$

Therefore,

$$\boxed{\mathbf{B}' = \mathbf{B} - \frac{\Phi}{2\pi}\left(\frac{\delta(r)}{r}\hat{\mathbf{z}}\right)}$$

6. Number 5

Space Charge Limited Flow

A vacuum tube is an example of a non-neutral plasma; current that flows through the tube (acting as a diode) is carried by electrons that are pulled from an emissive (hot) cathode (at ground) and accelerated across a gap to an anode (held at potential V_0). Current flow in the tube is *space charge limited*; in steady state, the space charge of the electron cloud between the anode and cathode reduces the electric field to zero at the cathode. Model the vacuum tube as two plates of area A, separated by distance d, with an electron cloud carrying a current of I between the two. You can treat the problem in 1-D (potential ϕ, charge density ρ, electron velocity v are all functions of x only with the coordinate axis normal to the two plates).

(a) Find an expression for the velocity of an electron between the plates given the potential $\phi(x)$. (assume the electrons start from rest at the cathode)

(b) Argue why the current I must be a constant (not a function of x) in steady state. Given this, determine the charge density as a function of position in the diode.

(c) Use the previous two results to write down a differential equation for the potential inside the diode, starting with Poisson's equation.

(d) Solve this equation for $\phi(x)$. Plot this and compare to the potential between the plates in vacuum.

(e) Show that $I = KV^{3/2}$, where V is the total potential difference across the two plates and K is a constant. This is the Child-Langmuir Law for space-charge limited current.

6.1. part (a)

$\frac{1}{2}mv^2 = q\phi(x) \rightarrow \boxed{v = \sqrt{\dfrac{2q\phi(x)}{m}}}$

6.2. part (b)

$\frac{\partial \rho}{\partial t} + \nabla \cdot \mathbf{j} = 0$

and since we know $\frac{\partial \rho}{\partial t} = 0$, this implies:

$\nabla \cdot \mathbf{j} = 0$

which then implies that current must be constant $\boxed{I = I_0}$.

6.3. part (c)

$\nabla^2 \phi = -\frac{\rho(x)}{\epsilon_0}$

and using the following result:

$\rho(x)\mathbf{v} = I/A$ where A is the cross-sectional area of the current, and I is the current magnitude.

$\nabla^2 \phi = -\frac{I}{A\mathbf{v}\epsilon_0} = -\frac{I}{A\epsilon_0}\sqrt{\frac{m}{2q\phi(x)}} = -\frac{I}{A\epsilon_0}\sqrt{\frac{m}{2q}}\phi(x)^{-1/2}$

$$\boxed{\frac{d^2\phi(x)}{dx^2} + \frac{I}{A\epsilon_0}\sqrt{\frac{m}{2q}}\phi(x)^{-1/2} = 0}$$

6.4. part (d)

Change of variables:

$\xi = \frac{d\phi}{dx}$

which implies

$\frac{d^2\phi}{dx^2} = \frac{d}{dx}\frac{d\phi}{dx} = \frac{d}{dx}\xi(\phi) = \frac{d\xi}{d\phi}\frac{d\phi}{dx} = \frac{d\xi}{d\phi}\xi(\phi)$

$\frac{d\xi}{d\phi}\xi(\phi) + \frac{I}{A\epsilon_0}\sqrt{\frac{m}{2q}}\phi(x)^{-1/2} = 0$

$\frac{d\xi}{d\phi}\xi(\phi) = -\sqrt{\frac{mI^2}{2qA^2\epsilon_0^2}}\phi(x)^{-1/2}$

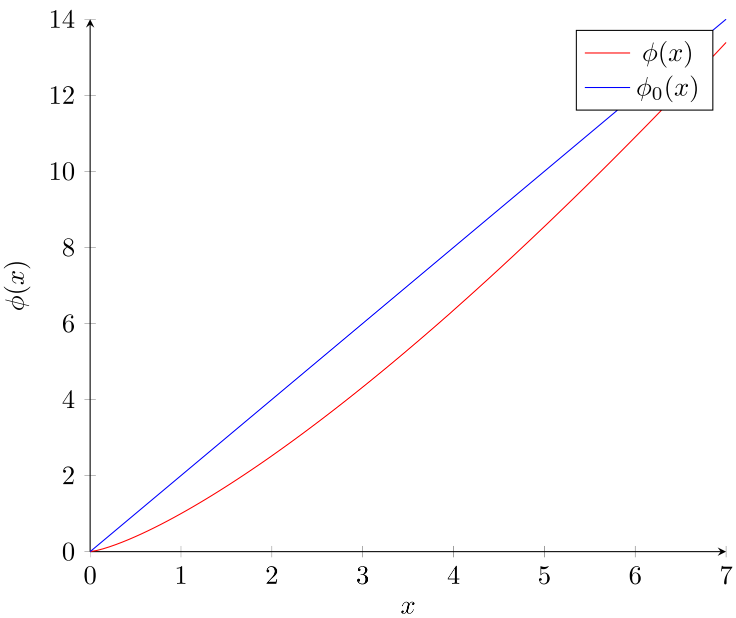
$\int \xi(\phi)d\xi = -\sqrt{\frac{mI^2}{2qA^2\epsilon_0^2}}\int \phi(x)^{-1/2}d\phi$

$\frac{1}{2}\xi(\phi)^2 = -\sqrt{\frac{mI^2}{2qA^2\epsilon_0^2}}\phi(x)^{1/2}$

$\xi(\phi) = \sqrt{-\sqrt{\frac{2mI^2}{qA^2\epsilon_0^2}}\phi(x)^{1/2}} = i\left(\frac{2mI^2}{qA^2\epsilon_0^2}\right)^{1/4}\phi(x)^{1/4} = i\gamma_0\phi(x)^{1/4} = \frac{d\phi}{dx}$ where I have defined γ_0 as $\left(\frac{2mI^2}{qA^2\epsilon_0^2}\right)^{1/4}$.

$\frac{-i}{\gamma_0}\phi(x)^{-1/4}d\phi = dx \to \frac{-4i}{3\gamma_0}\phi(x)^{3/4} = x \to \phi(x) = \left(\frac{3i\gamma_0}{4}\right)^{4/3}x^{4/3} = \gamma x^{4/3}$

$\boxed{\phi(x) = \gamma x^{4/3}}$ $\gamma = \left(\frac{3\gamma_0}{4}\right)^{4/3}$



Where the difference will truly show itself for large enough x.

6.5. part (e)

We are given that at $x = d$, $\phi = V_0$

$V_0 = \gamma x^{4/3} = \gamma d^{4/3} = \left(\frac{3}{4}\gamma_0\right)^{4/3}d^{4/3} = \left(\frac{3}{4}\left(\frac{2mI^2}{qA^2\epsilon_0^2}\right)^{1/4}\right)^{4/3}d^{4/3} = \left[\frac{3}{2}\frac{m}{qA^2\epsilon_0^2}d\right]^{4/3}I^{2/3} = AI^{2/3}$

where I have defined $A = \left[\frac{3}{2}\frac{m}{qA^2\epsilon_0^2}d\right]^{4/3}$
$V_0 = AI^{3/2}$
$I^{3/2} = V_0/A$
$I = \left(\frac{V_0}{A}\right)^{3/2} = \left(\frac{1}{A}\right)^{3/2}V_0^{3/2} = \boxed{KV_0^{3/2}}$, $K = A^{-3/2} = \left[\frac{3}{2}\frac{m}{qA^2\epsilon_0^2}d\right]^{-2}$

Submitted by Delano Campos on .

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— **Electromagnetic Theory: Magnetic Multipoles, and Magnetic Force and Energy**

1. 11.4

The Magnetic Moment of a Rotating Charged Disk

A compact disk with radius R and uniform surface charge density σ rotates with angular speed ω. Find the magnetic dipole moment m when the axis of rotation is

(a) the symmetry axis of the disk.

(b) any diameter of the disk.

1.1. Solution

1.1.1. part (a).

We start with

$$\mathbf{m} = \tfrac{1}{2}\int(\mathbf{r}\times\mathbf{K})da = \tfrac{1}{2}\int(r\hat{r}\times\sigma\mathbf{v})da = \tfrac{\sigma}{2}\int r(\hat{r}\times(\omega\times\mathbf{r}))da = \tfrac{\sigma\omega}{2}\int r^2(\hat{r}\times(\hat{z}\times\hat{r}))da$$

$$\tfrac{\sigma\omega}{2}\int r^2\left(\hat{r}\times\left(\left(\cancel{\cos\theta}\hat{r}-\sin\theta\hat{\theta}\right)\times\hat{r}\right)\right)da = \tfrac{\sigma\omega}{2}\int r^2\left(\hat{r}\times\hat{\phi}\right)r\sin(\pi/2)drd\phi = \tfrac{\sigma\omega}{2}\int r^3\left(-\hat{\theta}\right)drd\phi$$

$$= \tfrac{\sigma\omega}{2}\int -\left(\tfrac{\cancel{z}\hat{r}-r\hat{z}}{\sqrt{r^2+\cancel{z^2}}}\right)r^3drd\phi = \tfrac{\sigma\omega}{2}\int(\hat{z})r^3drd\phi = \tfrac{\sigma\omega}{2}\hat{z}\int_0^{2\pi}d\phi\int_0^R drr^3dr = \boxed{\frac{R^4\pi\sigma\omega}{4}}$$

1.1.2. part (b).

Likewise,

$$\mathbf{m} = \tfrac{1}{2}\int(\mathbf{r}\times\mathbf{K})da = \tfrac{1}{2}\int\omega\left[r^2\sin^2\phi\,\hat{x} - r^2\sin\phi\cos\phi\,\hat{y}\right]da = \boxed{-\frac{\pi}{8}\sigma R^4\omega}$$

2. 11.15

A Spherical Superconductor

A superconductor has the property that its interior has $B = 0$ under all conditions. Let a sphere (radius R) of this kind sit in a uniform magnetic field B_0.

(a) Place a fictitious point magnetic dipole m at the center of the sphere. Find m from the matching condition on the normal component of B.

(b) In reality, the dipole field in part (a) is created by a current density K which appears on the surface of the sphere. Find K from the matching condition on the tangential component of B.

(c) Confirm your answer in part (a) by computing the magnetic dipole moment associated with K from part (b).

2.1. Solution

2.1.1. part (a):

Here we use the normal component of the B field defined at the boundary to find $\mathbf{m}$.

$\mathbf{B_d} = \frac{\mu_0}{4\pi}\frac{3\hat{r}(\hat{r}\cdot\mathbf{m})-\mathbf{m}}{r^3}$ and $\mathbf{B_{ext}} = B_0\hat{z}$

$\hat{r}\cdot\left(\cancel{\mathbf{B}_1} - \mathbf{B}_2\right) = -\hat{r}\cdot(\mathbf{B_d}+\mathbf{B_{ext}}) = 0$

$-\hat{r}\cdot\left(B_0\hat{z} + \frac{\mu_0}{4\pi}\frac{3\hat{r}(\hat{r}\cdot\mathbf{m})-\mathbf{m}}{r^3}\right) = 0$

$\left(B_0\cos\theta + \frac{\mu_0}{4\pi}\frac{3\hat{r}(\hat{r}\cdot\mathbf{m})-\mathbf{m}}{r^3}\right) = \left(B_0\cos\theta + \frac{\mu_0}{4\pi}\frac{3(\hat{r}\cdot\mathbf{m})-\mathbf{m}\cdot\hat{r}}{r^3}\right) = \left(B_0\cos\theta + \frac{\mu_0}{2\pi}\frac{m\cos\alpha}{r^3}\right) = 0$

$m = -\frac{2\pi B_0 R^3}{\mu_0}\frac{\cos\theta}{\cos\alpha}$

and since $\mathbf{m} = m\hat{z} \longrightarrow \alpha = \theta$

$$\boxed{\mathbf{m} = \frac{-2\pi B_0 R^3}{\mu_0}\hat{z}}$$

2.1.2. part (b):

Using,

$-\hat{r}\times\left(\cancel{\mathbf{B}_1} - \mathbf{B}_2\right) = \mu_0\mathbf{K} \longrightarrow \mathbf{K} = \frac{1}{\mu_0}(\hat{r}\times\mathbf{B}_2)$

$\hat{r}\times\mathbf{B}_2 = \hat{r}\times\left(B_0\hat{z} + \frac{\mu_0}{4\pi}\frac{3\hat{r}(\hat{r}\cdot\mathbf{m})-\mathbf{m}}{r^3}\right) = B_0\hat{r}\times\hat{z} + \frac{\mu_0}{4\pi}\frac{3\hat{r}\times\hat{r}(\hat{r}\cdot\mathbf{m})-\hat{r}\times\mathbf{m}}{r^3} = B_0\hat{r}\times\hat{z} - \frac{m\mu_0}{4\pi}\frac{\hat{r}\times\hat{z}}{r^3}$

$= B_0\hat{r}\times\left(\cancel{\cos\theta\hat{r}} - \sin\theta\hat{\theta}\right) - \frac{m\mu_0}{4\pi}\frac{\hat{r}\times\left(\cancel{\cos\theta\hat{r}}-\sin\theta\hat{\theta}\right)}{r^3} = \frac{m\mu_0\sin\theta}{4\pi}\frac{\hat{r}\times\hat{\theta}}{r^3} - B_0\sin\theta\left(\hat{r}\times\hat{\theta}\right)$

$= \frac{m\mu_0\sin\theta}{4\pi}\frac{1}{r^3}\hat{\phi} - B_0\sin\theta\hat{\phi}$

$\mathbf{K} = \frac{1}{\mu_0}\left(\frac{m\mu_0\sin\theta}{4\pi}\frac{1}{r^3}\hat{\phi} - B_0\sin\theta\hat{\phi}\right) = \left[\frac{1}{4\pi R^3}\frac{2\pi B_0 R^3}{\mu_0} - \frac{B_0}{\mu_0}\right]\sin\theta\hat{\phi}$

$= -\left[\frac{B_0}{2\mu_0} + \frac{B_0}{\mu_0}\right]\sin\theta\hat{\phi} = \boxed{\frac{-3B_0}{2\mu_0}\sin\theta\hat{\phi}}$

2.1.3. part (c):

$\mathbf{m} = \frac{1}{2}\int(\mathbf{r}\times\mathbf{K})da = \frac{1}{2}\int R\hat{r}\times\left(-\frac{3B_0}{2\mu_0}\sin\theta\hat{\phi}\right)R^2\sin\theta d\theta d\phi = -\frac{3}{4}\frac{B_0}{\mu_0}R^3\int\left(\hat{r}\times\hat{\phi}\right)\sin^2\theta d\theta d\phi$

$= \frac{3}{4}\frac{B_0}{\mu_0}R^3\int\hat{\theta}\sin^2\theta d\theta d\phi = \frac{3}{4}\frac{B_0}{\mu_0}R^3\int(\cancel{\cos\theta\cos\phi\hat{x}} + \cancel{\cos\theta\sin\phi\hat{y}} - \sin\theta\hat{z})\sin^2\theta d\theta d\phi = -\frac{3}{4}\frac{B_0}{\mu_0}R^3\hat{z}\int\sin^3\theta d\theta d\phi$

$= -\frac{3}{4}\frac{B_0}{\mu_0}R^3 2\pi\hat{z}\int\sin^3\theta d\theta = -\frac{3}{4}\frac{B_0}{\mu_0}R^3 2\pi\left(\frac{4}{3}\right)\hat{z} = \boxed{\frac{-2\pi B_0}{\mu_0}R^3\hat{z}}$ Which is in agreement with part a.

3. 12.12

Three Point Dipoles

The diagram shows three small magnetic dipoles at the vertices of an equilateral triangle. Moments m_a and m_c point permanently along the internal angle bisectors. Moment m_b, is free to rotate in the plane of the triangle. Find the stable equilibrium orientation of the latter and the period of small oscillations around that orientation. Let all the moments have magnitude m and let m_b, have moment of inertia I about its center of mass.

3.1. Solution

Of course, considering small oscillations about a stable equilibrium implies we turn any $\cos\theta$ into $\cos\theta \approx 1 - \frac{1}{2}\theta^2$

Before I begin to slap down equations and plug in the results, I want to explicitly write out some of the results we will need, then evaluate them:

$$\begin{aligned}
\mathbf{m_a} \cdot \mathbf{m_b} &= m^2 \cos(\theta - \pi/3) \\
\mathbf{m_a} \cdot \mathbf{m_c} &= m^2 \cos(\theta + \pi/3) \\
\hat{r_c} \cdot \mathbf{m_c} &= m \cos(\pi/6) = m\sqrt{3}/2 \\
\hat{r_c} \cdot \mathbf{m_a} &= m \cos(\theta + \pi/6) \\
\hat{r_b} \cdot \mathbf{m_a} &= m \cos(\theta - \pi/6) \\
\hat{r_b} \cdot \mathbf{m_b} &= m \cos(\pi/6) = m\sqrt{3}/2
\end{aligned}$$

$V_a = -\mathbf{m_a}\cdot\mathbf{B}(a) = -\frac{\mu_0}{4\pi}\mathbf{m_a}\cdot[\mathbf{B_b} + \mathbf{B_c}] = -\frac{\mu_0}{4\pi a^3}[3\mathbf{m_a} \cdot \hat{r_b}(\hat{r_b} \cdot \mathbf{m_b}) - \mathbf{m_a} \cdot \mathbf{m_b} + 3\mathbf{m_a} \cdot \hat{r_c}(\hat{r_c} \cdot \mathbf{m_c}) - \mathbf{m_a} \cdot \mathbf{m_c}] =$

$= -\frac{\mu_0}{4\pi a^3}\left[3m\cos(\theta - \pi/6)\left(m\sqrt{3}/2\right) - m^2\cos(\theta - \pi/3) + 3m\cos(\theta + \pi/6)\left(m\sqrt{3}/2\right) - m^2\cos(\theta + \pi/3)\right]$

$= \frac{m^2\mu_0}{4\pi a^3}\left[\cos(\theta + \pi/3) + \cos(\theta - \pi/3) - 3\cos(\theta - \pi/6)\left(\sqrt{3}/2\right) - 3\cos(\theta + \pi/6)\left(\sqrt{3}/2\right)\right]$

Then I let mathematica evaluate the sum:

$V = \frac{-7\mu_0 m^2}{8\pi a^3}\cos\theta$

We can then set up the lagrangian to obtain the equation of motion, from which we can conclude the frequency of small oscillations:

$\mathcal{L} = T - V = \frac{1}{2}I\dot{\theta}^2 + \frac{7\mu_0 m^2}{8\pi a^3}\left(1 - \frac{1}{2}\theta^2\right)$

And evaluating the euler lagrange equations,

$\frac{\partial\mathcal{L}}{\partial q} - \frac{d}{dt}\frac{\partial\mathcal{L}}{\partial\dot{q}} = 0 \longrightarrow \boxed{\ddot{\theta} + \frac{7\mu_0 m^2}{8I\pi a^3}\theta = 0}$

Where we can define: $\omega^2 = \frac{7\mu_0 m^2}{8I\pi a^3} \longrightarrow \boxed{\omega = \sqrt{\frac{7\mu_0 m^2}{8I\pi a^3}}}$ as the frequency of small oscillations and

$T = \frac{2\pi}{\omega} = \boxed{2\pi\sqrt{\frac{8I\pi a^3}{7\mu_0 m^2}}}$ as the period of small oscillations.

4. 12.16

Magnetic Trap I

Two infinite, straight, parallel wires, each carrying current I in the same direction, are coincident with the lines $(1, 0, z)$ and $(-1, 0, z)$. In addition, there is a large external field $B_0 = B_0\hat{z}$. An atom (mass M) whose magnetic dipole moment m_0 is always anti-parallel to the local magnetic field direction can be trapped at the origin. What is the approximate frequency of small oscillations in the vicinity of the origin?

4.1. Solution

The problem as currently stated will only produce a contribution from the external field at exactly the origin. We get around this by Taylor expanding the total magnetic field about the origin, to order 3.

The total field is given by:

$\mathbf{B_{tot}} = \mathbf{B_{ext}} + \mathbf{B_1} + \mathbf{B_2} = B_0\hat{z} + \frac{\mu_0}{2\pi r}[I_1 + I_2]\hat{\phi} = B_0\hat{z} + \frac{\mu_0}{\pi r}I\hat{\phi} = B_0\hat{z} + \frac{\mu_0 I}{\pi}\left[\frac{-y\hat{x}+(x-1)\hat{y}}{(x-1)^2+y^2} + \frac{-y\hat{x}+(x+1)\hat{y}}{(x+1)^2+y^2}\right]$

$V = -\mathbf{m}\cdot\mathbf{B_{tot}} = -|\mathbf{m_0}||\mathbf{B}|\cos\pi = |m_0|\sqrt{B_x^2 + B_y^2 + B_z^2} \approx |m_0|\Big(B_0 + \frac{\mu_0 I}{B_0\pi}r^2\Big)$

Constructing the Lagrangian and then evaluating the Euler-Lagrange Equations:

$\mathcal{L} = T - V = \frac{1}{2}m\dot{\mathbf{r}}^2 - |m_0|\Big(B_0 + \frac{\mu_0 I}{B_0\pi}r^2\Big)$

$\frac{\partial\mathcal{L}}{\partial q} - \frac{d}{dt}\frac{\partial\mathcal{L}}{\partial\dot{q}} \longrightarrow -m\ddot{\mathbf{r}} - m_0\frac{2\mu_0 I}{B_0\pi}r = 0$ omitting third order r,

$\ddot{\mathbf{r}} + \frac{2m_0\mu_0 I}{mB_0\pi}r = 0$

Where I then define the frequency of small oscillations about the origin as:

$\omega^2 = \frac{2m_0\mu_0 I}{mB_0\pi} \longrightarrow \boxed{\omega = \sqrt{\frac{2m_0\mu_0 I}{mB_0\pi}}}$

Where it might serve useful to the reader to remind then that m_0 is NOT a mass, and the units work correctly to yield a frequency.

5. 13.13

Magnetic Shielding

A uniform external field $\mathbf{B}_{\text{ext}} = B_{\text{ext}}\hat{x}$ is applied to an infinitely long cylindrical shell with inside radius a, outside radius b, and relative permeability $\kappa = \mu/\mu_0$. The rest of space is vacuum. Show that the field inside the shell is screened to the value

$$\mathbf{B}_{in} = \frac{4\kappa b^2}{(\kappa+1)^2 b^2 - (\kappa-1)^2 a^2}\mathbf{B}_{ext}$$

where κ is a dimensionless parameter related to the geometry and material properties of the shell.

5.1. Solution

The problem boils down to utilizing the following boundary conditions, with the general solution to laplaces equation:

$$\psi_{cyl}(a) = \psi_{in}(a)$$
$$\psi_{cyl}(b) = \psi_{ext}(b)$$
$$\hat{\mathbf{n}}\cdot(H_1 - H_2) = 0 \longrightarrow \mu\frac{\partial\psi_{cyl}}{\partial n}|_a = \mu_0\frac{\partial\psi_{in}}{\partial n}|_a \longrightarrow \kappa\frac{\partial\psi_{cyl}}{\partial n}|_a = \frac{\partial\psi_{in}}{\partial n}|_a$$
$$\mu\frac{\partial\psi_{cyl}}{\partial n}|_b = \mu_0\frac{\partial\psi_{ext}}{\partial n}|_b \longrightarrow \kappa\frac{\partial\psi_{cyl}}{\partial n}|_b = \frac{\partial\psi_{ext}}{\partial n}|_b$$

The general solution to laplace's equation in polar coords:

$\psi(r,\phi) = (A_0 + B_0 ln(r))(x_0 + y_o\phi) + \sum_\alpha (A_\alpha r^\alpha + B_\alpha r^{-\alpha})(x_\alpha \sin\alpha\phi + y_\alpha \cos\alpha\phi)$

where $\alpha = 1$ and we eliminate all $\sin\alpha\phi$ terms since:

$\psi_{ext}(r \longrightarrow \infty) = \int -H_{ext} dx = \int -\frac{B_0}{\mu_0} dx = -\frac{B_0}{\mu_0} r\cos\phi$

which shows that $\alpha = 1$ and all $\sin\alpha\phi$ terms vanish. In addition, the first set of terms vanish as well since $k \neq 0$.

$\psi(r,\phi) = \left(Ar + Br^{-1}\right)\cos\phi$

employing boundary conditions from above:

$\psi_{in}(r = 0, \phi) = (Ar + \cancel{Br^{-1}})\cos\phi = Ar\cos\phi$

$\psi_{cyl}(r,\phi) = \left(Cr + Dr^{-1}\right)\cos\phi$

$\psi_{ext} = -\frac{B_0}{\mu_0} r\cos\phi = \left(Er + Fr^{-1}\right)\cos\phi$

$\psi_{ext}(r \longrightarrow \infty) = (Er + \cancel{Fr^{-1}})\cos\phi = -\frac{B_0}{\mu_0} r\cos\phi \longrightarrow \boxed{E = \frac{-B_0}{\mu_0} = -H_{ext}}$

$\psi_{in}(a) = \psi_{cyl}(a) \longrightarrow Aa\cos\phi = \left(Ca + Da^{-1}\right)\cos\phi \longrightarrow A = \left(C + Da^{-2}\right)$

$\psi_{ext}(b) = \psi_{cyl}(b) \longrightarrow -H_{ext} + Fb^{-2} = C + Db^{-2} \longrightarrow C = \frac{1}{b^2}(F - D) - H_{ext}$

$\kappa\frac{\partial\psi_{cyl}}{\partial n}|_a = \frac{\partial\psi_{in}}{\partial n}|_a \longrightarrow A = \kappa\left(C - Da^{-2}\right)$

$\kappa\frac{\partial\psi_{cyl}}{\partial n}|_b = \frac{\partial\psi_{ext}}{\partial n}|_b \longrightarrow \kappa\left(C - Db^{-2}\right) = E + Fb^{-2} \longrightarrow F$

Solving the system:

$$A = \frac{4H_{\text{ext}}b^2\kappa}{a^2(\kappa-1)^2 - b^2(\kappa+1)^2}$$

$$C = \frac{2H_{\text{ext}}b^2(\kappa+1)}{a^2(\kappa-1)^2 - b^2(\kappa+1)^2}$$

$$D = \frac{2H_{\text{ext}}a^2b^2(\kappa-1)}{a^2(\kappa-1)^2 - b^2(\kappa+1)^2}$$

$$F = \frac{H_{\text{ext}}b^2(b^2-a^2)(\kappa^2-1)}{b^2(\kappa+1)^2 - a^2(\kappa-1)^2}$$

At which point we can find H from ψ, then find B from H:

$\mathbf{H_{in}} = -\nabla\psi_{in} = H_{\text{in}} = \frac{4H_{\text{ext}}b^2\kappa}{b^2(\kappa+1)^2 - a^2(\kappa-1)^2}\left(\cos\phi\hat{\rho} - \sin\phi\hat{\phi}\right)$

$$\boxed{\mathbf{B_{in}} = \frac{4\kappa b^2}{(\kappa+1)^2b^2 - (\kappa-1)^2a^2}B_{\text{ext}}}$$

Submitted by Delano Campos on .

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— **Electromagnetic Theory II:**

1. Jackson 7.6

A plane wave of frequency ω is incident normally from vacuum on a semi-infinite slab of material with a *complex* index of refraction $n(\omega)$ $[n^2(\omega) = \epsilon(\omega)/\epsilon_0]$.

(a) Show that the ratio of reflected power to incident power is

$$R = \left|\frac{1-n(\omega)}{1+n(\omega)}\right|^2$$

while the ratio of power transmitted into the medium to the incident power is

$$T = \frac{4\,\mathrm{Re}\,n(\omega)}{|1+n(\omega)|^2}$$

(b) Evaluate $\mathrm{Re}\,\{i\omega(\mathbf{E}\cdot\mathbf{D}^* - \mathbf{B}\cdot\mathbf{H}^*)/2\}$ as a function of (x, y, z). Show that this rate of change of energy per unit volume accounts for the relative transmitted power T.

(c) For a conductor, with $n^2 = 1 + i(\sigma/\omega\epsilon_0)$, or real, write out the results of parts a and b in the limit $\epsilon_0\omega \ll \sigma$. Express your answer in terms of δ as much as possible. Calculate $\frac{1}{2}\,\mathrm{Re}(\mathbf{J}^*\cdot\mathbf{E})$ and compare with the result of part b. Do both enter the complex form of Poynting's theorem?

1.1. solution

(a) Here I start the derivation of R from the poynting vector:
$R = \frac{\langle\mathbf{S}''\rangle\cdot(-\hat{z})}{\langle\mathbf{S}\rangle\cdot(\hat{z})} = \frac{\mathrm{Re}\left[\mathbf{E}''\times\mathbf{H}''^*\right]}{\mathrm{Re}[\mathbf{E}\times\mathbf{H}^*]} = \left|\frac{E_0''}{E_0}\right|^2$

to which then we need the reflected wave amplitude as a function of the incident. Here I use the plane wave result as solutions to maxwell's wave equations, then make use of boundary conditions at the interface. I make the assumption here that $\mu = \mu'$ and make use of the fact that $\mathbf{B} = \frac{\mathbf{k}\times\mathbf{E}}{\omega} = \frac{kE}{\omega} = \frac{n(\omega)E}{c}$
$E_0 - E_0'' = E_0'$
$\frac{1}{\mu}[B_0 + B_0''] = \frac{1}{\mu'}[B_0'] \Longrightarrow n(\omega)(E_0 + E_0'') = n'(\omega)E_0'$

using these gives us:
$E_0'' + \frac{n'(\omega)}{n(\omega)}E_0'' = E_0\left(\frac{n'(\omega)}{n(\omega)} - 1\right) \Longrightarrow E_0'' = \frac{\frac{n'(\omega)}{n(\omega)}-1}{\frac{n'(\omega)}{n(\omega)}+1}E_0 = \frac{n'(\omega)-n(\omega)}{n'(\omega)+n(\omega)}E_0$

Now we are in a position to find this ratio:
$R = \left|\frac{E_0''}{E_0}\right|^2 = \left|\frac{\frac{n'(\omega)-n(\omega)}{n'(\omega)+n(\omega)}E_0}{E_0}\right|^2 = \left|\frac{n'(\omega)-n(\omega)}{n'(\omega)+n(\omega)}\right|^2$

and since vacuum:

$$\boxed{R = \left|\frac{1-n(\omega)}{1+n(\omega)}\right|^2}$$

and to find transmission we use conservation of energy:
$T = 1-R = 1-\left|\frac{1-n(\omega)}{1+n(\omega)}\right|^2 = \frac{|1+n(\omega)|^2-|1-n(\omega)|^2}{|1+n(\omega)|^2} = \frac{\mathbf{Re}\left\{(1+n(\omega))^2-(1-n(\omega))^2\right\}}{|1+n(\omega)|^2} = \frac{\mathbf{Re}\left\{1+2n(\omega)+n(\omega)^2-1+2n(\omega)-n(\omega)^2\right\}}{|1+n(\omega)|^2}$

Which is then clearly what we wanted:

$$\boxed{T = \frac{4\mathbf{Re}[n(\omega)]}{|1+n(\omega)|^2}}$$

(b) If our goal is the transmission coefficient, we need the following expression:
$T = \frac{P_T/A}{P_I/A} = \frac{\langle S_T\rangle}{\langle S_I\rangle}$

But the trick here is to integrate the expression we were given over the space that the wave propogates. I make a dimensional argument for this: if the given expression is [energy/vol ·time], this is just [power/vol] = [(power/area) · 1/length] where when I integrate the expression over a length distance, we recover an intensity as [power/area], which is the Poynting vector.
$\mathbf{Re}\left[\frac{i\omega}{2}(\mathbf{E}\cdot\mathbf{D}^* - \mathbf{B}\cdot\mathbf{H}^*)\right] = \mathbf{Re}\left[\frac{i\omega}{2}\left(\epsilon^*(\omega)\mathbf{E}\cdot\mathbf{E}^* - \frac{1}{\mu_0}\mathbf{B}\cdot\mathbf{B}^*\right)\right] = \mathbf{Re}\left[\frac{i\omega}{2}\left(\epsilon^*(\omega)|\mathbf{E}|^2 - \frac{1}{\mu_0}|\mathbf{B}|^2\right)\right]$
$= \mathbf{Re}\left[\frac{i\omega}{2}\left(\epsilon^*(\omega)|\mathbf{E}|^2 - \frac{1}{\mu_0}\left|\frac{n(\omega)}{c}\right|^2|\mathbf{E}|^2\right)\right] = \mathbf{Re}\left[\frac{i\omega}{2}\left(\epsilon^*(\omega)|\mathbf{E}|^2 - |n(\omega)|^2\epsilon_0|\mathbf{E}|^2\right)\right]$

where here I will use the relation between the frequency dependent index of refraction and the permittivity of the material:
$\epsilon^*(\omega) = n^{*2}(\omega)\epsilon_0$

$= \mathbf{Re}\left[\frac{i\omega}{2}\left(n^{*2}(\omega)\epsilon_0|\mathbf{E}|^2 - |n(\omega)|^2\epsilon_0|\mathbf{E}|^2\right)\right] = \mathbf{Re}\left[\frac{i\omega}{2}\left(n^{*2}(\omega) - |n(\omega)|^2\right)\epsilon_0|\mathbf{E}|^2\right]$

Next I make use of the tool we used in class to split the wave vector into a real and imaginary component of the electric field since our exponential for plane waves is purely imaginary:
$= \mathbf{Re}\left[\frac{i\omega\epsilon_0}{2}\left(n^{*2}(\omega) - |n(\omega)|^2\right)|\mathbf{E_0}|^2 e^{2\ \mathbf{Im}[\mathbf{k}(\omega)\cdot\mathbf{r}]}\right] = \mathbf{Re}\left[\frac{i\omega\epsilon_0}{2}\left(n^{*2}(\omega) - |n(\omega)|^2\right)|\mathbf{E_0}|^2 e^{2\ \mathbf{Im}[\mathbf{k}(\omega)\cdot\mathbf{r}]}\right]$
$= \frac{\omega\epsilon_0}{2}(-2\mathbf{Re}[n(\omega)]\mathbf{Im}[n(\omega)])\ \mathbf{Re}\left[|\mathbf{E_0}|^2\right]e^{2\ \mathbf{Im}[\mathbf{k}(\omega)\cdot\mathbf{r}]} = -\omega\epsilon_0\mathbf{Re}[n(\omega)]\mathbf{Im}[n(\omega)]\ \mathbf{Re}\left[|\mathbf{E_0}|^2\right]e^{2\ \mathbf{Im}[\mathbf{k}(\omega)\cdot\mathbf{r}]}$
$= -\omega\epsilon_0\mathbf{Re}[n(\omega)]\mathbf{Im}\left[\frac{1}{\sqrt{\mu_0\epsilon_0}\omega}k(\omega)\right]\ \mathbf{Re}\left[|\mathbf{E_0}|^2\right]e^{2\ \mathbf{Im}[\mathbf{k}(\omega)\cdot\mathbf{r}]}$
$= -\sqrt{\frac{\epsilon_0}{\mu_0}}\mathbf{Re}[n(\omega)]\mathbf{Im}[k(\omega)]\ \mathbf{Re}\left[|\mathbf{E_0}|^2\right]e^{2\ \mathbf{Im}[\mathbf{k}(\omega)\cdot\mathbf{r}]}$

Assuming the wave propagates in z starting at the origin where the medium begins:
$\int_0^\infty \mathbf{Re}\left[\frac{i\omega}{2}(\mathbf{E}\cdot\mathbf{D}^* - \mathbf{B}\cdot\mathbf{H}^*)\right]dz = \int_0^\infty -\sqrt{\frac{\epsilon_0}{\mu_0}}\mathbf{Re}[n(\omega)]\mathbf{Im}[\mathbf{k}(\omega)]\ \mathbf{Re}\left[|\mathbf{E_0}|^2\right]e^{2\ \mathbf{Im}[k(\omega)]\mathbf{r}\cdot\hat{k}}dz$
$= -\frac{1}{2}\sqrt{\frac{\epsilon_0}{\mu_0}}\mathbf{Re}[n(\omega)]|\mathbf{E_0}|^2$

$$T = \frac{-\frac{1}{2}\sqrt{\frac{\epsilon_0}{\mu_0}}\mathbf{Re}[n(\omega)]|\mathbf{E_0}|^2}{-\frac{1}{2}\sqrt{\frac{\epsilon_0}{\mu_0}}|\mathbf{E_0'}|^2} = \mathbf{Re}[n(\omega)]\frac{|\mathbf{E_0}|^2}{|\mathbf{E_0'}|^2} = \boxed{\mathbf{Re}[n(\omega)]\frac{4}{|1+n(\omega)|^2}}$$

(c) Since the results of a and b are identical, with only the approach differing, I first write them out using the skin depth and in the limit $\sigma >> \epsilon_0\omega$, we need to somehow find space for the skin depth to show up. We achieve this by re writing the index of refraction:
$R = \left|\frac{1-n(\omega)}{1+n(\omega)}\right|^2 = \left|\frac{1-\sqrt{1+i(\sigma/\omega\epsilon_0)}}{1+\sqrt{1+i(\sigma/\omega\epsilon_0)}}\right|^2 = \left|\frac{1-(1+i)\sqrt{(\sigma/2\omega\epsilon_0)}}{1+(1+i)\sqrt{(\sigma/2\omega\epsilon_0)}}\right|^2 = \left|\frac{1-(1+i)\frac{c}{\omega\delta}}{1+(1+i)\frac{c}{\omega\delta}}\right|^2 = \left|\frac{1-(1-i)\frac{2c}{\omega\delta}}{1-(1+i)\frac{2c}{\omega\delta}}\right|^2 \approx$
$\boxed{1-\frac{2\omega\delta}{c}}$

Knowing R,

$T = 1-R \approx 1-1+\frac{2\omega\delta}{c} \approx \boxed{\frac{2\omega\delta}{c}}$

Next the question asks to calculate $\frac{1}{2}\,\mathrm{Re}(\mathbf{J}^* \cdot \mathbf{E})$ and compare this result to part b, and to answer whether or not both expressions are in the complex form of the Poynting theorem:

$\frac{1}{2}\,\mathrm{Re}(\mathbf{J}^* \cdot \mathbf{E}) = \frac{1}{2}\,\mathrm{Re}\{\sigma^* \mathbf{E}^* \cdot \mathbf{E}\} = \frac{1}{2}\sigma\,\mathrm{Re}\Big\{|\mathbf{E}|^2\Big\} = \frac{1}{2}\sigma\,\mathrm{Re}\Big\{|\mathbf{E}|^2\Big\}$

and since, $\sqrt{\frac{\sigma}{2\omega\epsilon_0}} = \frac{c}{\omega\delta} \Longrightarrow \sigma = 2\omega\epsilon_0\big(\frac{c}{\omega\delta}\big)^2$

$\frac{1}{2}\sigma\,\mathrm{Re}\Big\{|\mathbf{E}|^2\Big\} = \frac{1}{2}2\omega\epsilon_0\big(\frac{c}{\omega\delta}\big)^2\,\mathrm{Re}\Big\{|\mathbf{E}|^2\Big\} = \frac{\epsilon_0 c^2}{\omega\delta^2}|E_0|^2 e^{-2\,\mathrm{Im}\{\mathbf{k}(\omega)\cdot\mathbf{r}\}}$

$k(\omega)/\omega = n(\omega)/c \Longrightarrow k(\omega) = n(\omega)\omega/c$

$\frac{1}{2}\sigma\,\mathrm{Re}\Big\{|\mathbf{E}|^2\Big\} = \frac{\epsilon_0 c^2}{\omega\delta^2}|E_0|^2 e^{-2\,\mathrm{Im}\big\{(n(\omega)\omega/c)\hat{k}\cdot\mathbf{r}\big\}} = \boxed{\dfrac{\epsilon_0 c^2}{\omega\delta^2}|E_0|^2 e^{-2/\delta \hat{k}\cdot\mathbf{r}}}$

which yields an identical result to computing what we did in part b:

$\mathbf{Re}\big[\frac{i\omega}{2}(\mathbf{E}\cdot\mathbf{D}^* - \mathbf{B}\cdot\mathbf{H}^*)\big] = \frac{1}{2}\sqrt{\frac{\epsilon_0}{\mu_0}}\mathbf{Re}[n(\omega)]|\mathbf{E_0}|^2 \approx \epsilon_0\omega\big(\frac{c}{\omega\delta}\big)^2|E_0|^2 e^{-2/\delta\hat{k}\cdot\mathbf{r}} = \boxed{\dfrac{\epsilon_0 c^2}{\omega\delta^2}|E_0|^2 e^{-2/\delta\hat{k}\cdot\mathbf{r}}}$

which is enough to make an argument as to why adding both of these expressions into one theorem would be useless, since we would have accounted for the qunatity twice. so no, they both do not show up.

2. Jackson 7.13

A stylized model of the ionosphere is a medium described by the dielectric constant (7.59). Consider the earth with such a medium beginning suddenly at a height h and extending to infinity. For waves with polarization both perpendicular to the plane of incidence (from a horizontal antenna) and in the plane of incidence (from a vertical antenna),

(a) show from Fresnel's equations for reflection and refraction that for $\omega > \omega_p$, there is a range of angles of incidence for which reflection is not total, but for larger angles there is total reflection back toward the earth.

(b) A radio amateur operating at a wavelength of 21 meters in the early evening finds that she can receive distant stations located more than 1000 km away, but none closer. Assuming that the signals are being reflected from the F layer of the ionosphere at an effective height of 300 km, calculate the electron density. Compare with the known maximum and minimum F layer densities of $\sim 2\times 10^{12}\,\mathrm{m}^{-3}$ in the daytime and $\sim (2-4)\times 10^{11}\,\mathrm{m}^{-3}$ at night.

2.1. solution

(a) The point of this problem seemed to be that we can recover a simple result for the concept of the critical angle, except from a more general approach using Fresnel's equations. In both cases for perpendicular and parallel polarization to the plane of incidence, we get the same result, which is identical to the critical angle from snells law. Please comment on this if I missed the point of this part. This portion was effectively done in class when discussing the critical angle. In the high-frequency limit, we say,
$\frac{\epsilon(\omega)}{\epsilon_0} \approx 1 - \frac{\omega_p^2}{\omega^2}$
$n' = \sqrt{1 - \frac{\omega_p^2}{\omega^2}} = \frac{1}{\omega}\sqrt{\omega^2 - \omega_p^2} < 1$ and $n = 1$

and by Snell's law on the condition between the indices of refraction, we can conclude a critical angle such that $\theta' = \frac{\pi}{2}$
$n\sin\theta = n'\sin\theta' \Longrightarrow \theta_c = \sin^{-1}\left(\frac{n'}{n}\right)$

Using Fresnel's equations for the perpendicular case for reflection amplitudes:
$\frac{E_0''}{E_0} = \frac{n\cos\theta_i - \sqrt{n'^2 - n^2\sin^2\theta_i}}{n\cos\theta_i + \sqrt{n'^2 - n^2\sin^2\theta_i}} = \frac{\cos\theta_i - \sqrt{n'^2 - \sin^2\theta_i}}{\cos\theta_i + \sqrt{n'^2 - \sin^2\theta_i}} = \frac{\cos\theta_i - \sqrt{n'^2 - \sin^2\theta_i}}{\cos\theta_i + \sqrt{n'^2 - \sin^2\theta_i}}$

since we are after the threshold angle where total reflection occurs, we say that the portion of the square root has to either be zero or imaginary to give back a reflected to transmitted ratio of 1:
$-\sqrt{n'^2 - \sin^2\theta_i} = 0$
$n'^2 = \sin^2\theta_i \implies \theta_i = \sin^{-1} n' = \sin^{-1}\left(\frac{1}{\omega}\sqrt{\omega^2 - \omega_p^2}\right)$

Likewise for the parallel case, I state Fresnel's equation and deduce a condition on the incident angle (implicit assumption on permeabilities of the materials being approximately equal):
$\frac{E_0''}{E_0} = \frac{n'^2\cos\theta_i - n\sqrt{n'^2 - n^2\sin^2\theta_i}}{n'^2\cos\theta_i + n\sqrt{n'^2 - n^2\sin^2\theta_i}} = \frac{n'^2\cos\theta_i - \sqrt{n'^2 - \sin^2\theta_i}}{n'^2\cos\theta_i + \sqrt{n'^2 - \sin^2\theta_i}}$

Where it then becomes clear that we get an identical result when forcing the square root to vanish.

Therefore any incident angle below this threshold will give partial reflection, up until the point it is reach where we observe full reflection, contingent on $\omega > \omega_p$:

$$\boxed{\theta_c = \sin^{-1}\left(\frac{1}{\omega}\sqrt{\omega^2 - \omega_p^2}\right)}$$

(b) Here ultimately we want to make use of our knowledge about total reflection to find the the plasma frequency of the ionosphere, which will thus allow us to find a density of it:

$\omega_p^2 = \frac{NZe^2}{\epsilon_0 m} = \frac{n_v e^2}{\epsilon_0 m} \implies n_v = \frac{\epsilon_0 m\omega_p^2}{e^2}$

Where it then becomes obvious that we need to know the plasma frequency. Lets make use of part a information on critical angles and total reflection:
$\sin\theta_c = \frac{z}{\sqrt{4z_1 + z^2}} = \frac{1}{\omega}\sqrt{\omega^2 - \omega_p^2} \implies \omega_p = \omega\sqrt{\frac{4z_1}{4z_1+z}} = 2\pi f\sqrt{\frac{4z_1}{4z_1+z}} = 2\pi\frac{c}{\lambda}\sqrt{\frac{4z_1}{4z_1+z}} = 2\pi\frac{3\cdot 10^8\text{m/s}}{21\text{m}}\sqrt{\frac{4\cdot 300\text{km}}{4\cdot 300\text{km} + 1000\text{km}}}$

$\omega_p = 4.5 \cdot 10^7\text{Hz}$

$n_v = \frac{\epsilon_0 m\omega_p^2}{e^2} = \frac{8.85\cdot 10^{-12}\text{SI}\ 9.1\cdot 10^{-31}\text{kg}\ 4.5\cdot 10^{14}\text{Hz}}{1.6\cdot 10^{-38}\text{C}} = \boxed{2.5\cdot 10^{11}\text{m}^{-3}}$

3. Jackson 7.22

Use the Kramers–Kronig relation (7.120) to calculate the real part of $\epsilon(\omega)$, given the imaginary part of $\epsilon(\omega)$ for positive ω as

(a) $\text{Im}\left[\frac{\epsilon}{\epsilon_0}\right] = \lambda[\theta(\omega - \omega_1) - \theta(\omega - \omega_2)], \quad \omega_2 > \omega_1 > 0$

3.1. solution

(a) from Kramers-Kronig, we can relate imaginary and real components of a complex permittivity as such:

$\epsilon(\omega) = \epsilon_1(\omega) + i\epsilon_2(\omega)$
$\mathbf{Re}[\epsilon(\omega)/\epsilon_0] = 1 + \frac{2}{\pi}P\left[\int_0^\infty \frac{\omega'\mathbf{Im}[\epsilon(\omega)/\epsilon_0]}{\omega'^2 - \omega^2}d\omega'\right] = 1 + \frac{2}{\pi}P\left[\int_0^\infty \frac{\omega'\lambda[\theta(\omega-\omega_1) - \theta(\omega-\omega_2)]}{\omega'^2 - \omega^2}d\omega'\right]$
$= 1 + \frac{2\lambda}{\pi}P\left[\int_0^\infty \frac{\omega'\theta(\omega-\omega_1)}{\omega'^2 - \omega^2}d\omega' - \int_0^\infty \frac{\omega'\theta(\omega-\omega_2)}{\omega'^2 - \omega^2}d\omega'\right]$

And since the Heaviside step function is zero everywhere in frequency space that is not ω_1 and ω_2 respectively, we can adjust the bounds of integration since the integral will vanish everywhere that is not in the range (ω_i, ∞)

$$= 1 + \frac{2\lambda}{\pi} P\left[\int_{\omega_1}^{\infty} \frac{\omega'}{\omega'^2 - \omega^2} d\omega' - \int_{\omega_2}^{\infty} \frac{\omega'}{\omega'^2 - \omega^2} d\omega'\right] = 1 + \frac{\lambda}{\pi}\left[\ln\left(\left|\omega_2^2 - \omega^2\right|\right) - \ln\left(\left|\omega_1^2 - \omega^2\right|\right)\right]$$

Where we then recover what is asked of us:

$$\boxed{\mathbf{Re}[\epsilon(\omega)/\epsilon_0] = 1 + \frac{\lambda}{\pi}\left[\ln\left(\left|\omega_2^2 - \omega^2\right|\right) - \ln\left(\left|\omega_1^2 - \omega^2\right|\right)\right]}$$

Submitted by Delano Campos on .