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HOMEWORK 4 — Time-Dependant Perturbation Theory

1. 5.29

Consider a one-dimensional simple harmonic oscillator whose classical angular frequency is ω_0. For $t < 0$ it is known to be in the ground state. For $t > 0$ there is also a time-dependent potential

$$V(t) = F_0 x \cos \omega t$$

where F_0 is constant in both space and time. Obtain an expression for the expectation value $\langle x \rangle$ as a function of time using time-dependent perturbation theory to lowest non vanishing order. Is this procedure valid for $\omega \approx \omega_0$?

1.1. Solution

We are after the usual, $\langle x|x \rangle = \langle \phi | x | \phi \rangle_s$ where I make it a point that this is in the typical Schrodinger picture, since the distinction between pictures is now important enough to note it.
$\delta_{ni} = \delta_{n0} = 0$

$c_n^{(1)}(t) = \frac{-i}{\hbar} \int_{t_0}^{t} \langle n| V_I(t') |i\rangle \, dt' = \frac{-i}{\hbar} F_0 \int_{t_0}^{t} \langle n| x \cos(\omega t') |i\rangle \, e^{i(E_n - E_0)t'/\hbar} dt'$

$= \frac{-i}{\hbar} F_0 \sqrt{\frac{\hbar}{2m\omega_0}} \int_0^t \langle n| (a + a^\dagger) |i\rangle e^{i(E_n-E_0)t'/\hbar} \cos(\omega t') dt' = \frac{-i}{\hbar} F_0 \sqrt{\frac{\hbar}{2m\omega_0}} \int_0^t \langle n| (a^\dagger) |0\rangle e^{i(E_n-E_0)t'/\hbar} \cos(\omega t') dt'$

$= \frac{-i}{\hbar} F_0 \sqrt{\frac{\hbar}{2m\omega_0}} \int_0^t \langle n| a^\dagger |0\rangle e^{i(E_n-E_0)t'/\hbar} \cos(\omega t') dt' = \frac{-i}{\hbar} F_0 \sqrt{\frac{\hbar}{2m\omega_0}} \delta_{n1} \int_0^t e^{i(\omega_n - \omega_0)t'} (\frac{e^{i\omega t'} + e^{-i\omega t'}}{2}) dt'$

$= \frac{-i}{2\hbar} F_0 \sqrt{\frac{\hbar}{2m\omega_0}} \delta_{n1} \int_0^t e^{i(\omega_n - \omega_0 + \omega)t'} + e^{i(\omega_n - \omega_0 - \omega)t'} dt' = F_0 \sqrt{\frac{2}{\hbar m\omega_0}} \delta_{n1} [\frac{1}{\omega_n - \omega_0 + \omega} (e^{i(\omega_n - \omega_0 + \omega)t} - 1) +$
$\frac{1}{\omega_n - \omega_0 - \omega} (e^{i(\omega_n - \omega_0 - \omega)t} - 1)]$

and since $n = 1$ is the only allowed transition state:

$$\boxed{c_1^{(1)}(t) = F_0 \sqrt{\frac{2}{\hbar m\omega_0}} \left[\frac{e^{i(\omega_1 - \omega_0 + \omega)t} - 1}{\omega_1 - \omega_0 + \omega} + \frac{e^{i(\omega_1 - \omega_0 - \omega)t} - 1}{\omega_1 - \omega_0 - \omega} \right]}$$

$\langle x \rangle = \langle \alpha(t)| x |\alpha(t)\rangle_s = \sqrt{\frac{\hbar}{2m\omega_0}} \left[\langle 1| c_1^*(t) e^{iE_1 t/\hbar} + \langle 0| e^{iE_0 t/\hbar} \right] \left[c_1(t) e^{-iE_1 t/\hbar} |0\rangle + e^{-iE_0 t/\hbar} + \sqrt{2} c_1(t) e^{-iE_1 t/\hbar} |2\rangle \right]$

$= F_0 \sqrt{\frac{2}{\hbar m\omega_0}} \sqrt{\frac{\hbar}{2m\omega_0}} c_1^*(t) \left[e^{i(E_1 - E_0)t/\hbar} + e^{i(E_0 - E_1)t/\hbar} \right] = \frac{F_0}{(m\omega_0)^2} \left[\frac{e^{i(\omega_1 - \omega_0 + \omega)t} - 1}{\omega_1 - \omega_0 + \omega} + \frac{e^{i(\omega_1 - \omega_0 - \omega)t} - 1}{\omega_1 - \omega_0 - \omega} \right] \left[e^{i\bar{\omega}t} + e^{-i\bar{\omega}t} \right]$

Where I have defined $\bar{\omega}$ as $\bar{\omega} = (E_1 - E_0)/\hbar$

$= \frac{2F_0 \cos\bar{\omega}t}{(m\omega_0)^2} \left[\frac{e^{i(\omega_1 - \omega_0 + \omega)t} - 1}{\omega_1 - \omega_0 + \omega} + \frac{e^{i(\omega_1 - \omega_0 - \omega)t} - 1}{\omega_1 - \omega_0 - \omega} \right]$

$$\boxed{\langle x \rangle = \frac{2F_0 \cos\bar{\omega}t}{(m\omega_0)^2} \left[\frac{e^{i(\omega_1 - \omega_0 + \omega)t} - 1}{\omega_1 - \omega_0 + \omega} + \frac{e^{i(\omega_1 - \omega_0 - \omega)t} - 1}{\omega_1 - \omega_0 - \omega} \right]}$$

No this is not valid at resonance. We get an asymptote to infinity at resonance, thus voiding the usefulness of time-dependant perturbation theory here.

2. 5.31

Consider a particle bound in a simple harmonic oscillator potential. Initially ($t < 0$), it is in the ground state. At $t = 0$ a perturbation of the form

$$H'(x,t) = Ax^2 e^{-t/\tau}$$

is switched on. Using time-dependent perturbation theory, calculate the probability that, after a sufficiently long time ($t \gg \tau$), the system will have made a transition to a given excited state. Consider all final states.

2.1. Solution

$c_n^{(1)}(t) = \frac{-i}{\hbar}\int_{t_0}^{t} \langle n| V_I(t') |i\rangle\, dt' = \frac{-i}{\hbar} A \int_{t_0}^{t} \langle n| x^2 e^{-t'/\tau} |i\rangle\, e^{i(E_n - E_0)t'/\hbar} dt'$

$= \frac{-iA}{2m\omega}\int_{t_0}^{t} \langle n| (a + a^\dagger)^2 e^{-t'/\tau} |i\rangle\, e^{i(\omega_n - \omega_0)t'} dt'$

$= \frac{-iA}{2m\omega}\int_{t_0}^{t} \langle n| (aa + aa^\dagger + a^\dagger a + a^\dagger a^\dagger) |0\rangle\, e^{i(\omega_n - \omega_0 - 1/\tau)t'} = \frac{-iA}{2m\omega}(\delta_{n0} + \sqrt{2}\delta_{n2}) \int_{0}^{t} e^{i(\omega_n - \omega_0 - 1/\tau)t'} dt'$

where we gain new insight into the allowed transitions from the Kronecker deltas that pop out. Namely, 0 and 2.

$$\boxed{c_n^{(1)}(t) = \frac{-A}{2m\omega}(\delta_{n0} + \sqrt{2}\delta_{n2}) \left[\frac{1}{(\omega_n - \omega_0 - 1/\tau)}(e^{(\omega_n - \omega_0 - 1/\tau)t} - 1)\right]}$$

Where I then drop the kroneckers and evaluate each probability individually:

$|c_0^{(1)}(t)|^2 = \left|\frac{-A}{2m\omega}\left[\frac{1}{(\omega_0 - \omega_0 - 1/\tau)}(e^{(\omega_0 - \omega_0 - 1/\tau)t} - 1)\right]\right|^2 = \left(\frac{A\tau}{2m\omega}\right)^2 \left|\left[(e^{(\omega_0 - \omega_0 - 1/\tau)t} - 1)\right]\right|^2$

$= \text{lím}_{t\to\infty}\left(\left(\frac{A\tau}{2m\omega}\right)^2 \left|\left[(e^{(\omega_0 - \omega_0 - 1/\tau)t} - 1)\right]\right|^2\right) = \left(\frac{A\tau}{2m\omega}\right)^2 |(-1)|^2 = \boxed{\frac{1}{4}\left(\frac{A\tau}{m\omega}\right)^2}$

$|c_2^{(1)}(t)|^2 = \left|\frac{-A}{2m\omega}\left[\frac{\sqrt{2}}{(\omega_2 - \omega_0 - 1/\tau)}(e^{(\omega_2 - \omega_0 - 1/\tau)t} - 1)\right]\right|^2 = \left(\frac{\sqrt{2}A}{2m\omega}\right)^2 \frac{1}{(\omega_2 - \omega_0 - 1/\tau)^2} \left|\left[(e^{(\omega_2 - \omega_0 - 1/\tau)t} - 1)\right]\right|^2$

$= \text{lím}_{t\to\infty}\left(\left(\frac{\sqrt{2}A}{2m\omega(\omega_2 - \omega_0 - 1/\tau)}\right)^2 \left|\left[(e^{(\omega_2 - \omega_0 - 1/\tau)t} - 1)\right]\right|^2\right) = \left(\frac{\sqrt{2}A}{2m\omega(\omega_2 - \omega_0 - 1/\tau)}\right)^2 |(-1)|^2 = \boxed{\frac{1}{2}\left(\frac{A}{m\omega}\right)^2 \frac{1}{(\omega_2 - \omega_0 - 1/\tau)^2}}$

Where we have considered all final states possible.

3. 5.34

Consider a particle in one dimension moving under the influence of some time-independent potential. The energy levels and the corresponding eigenfunctions for this problem are assumed to be known. We now subject the particle to a traveling pulse represented by a time-dependent potential,

$$V(t) = A\delta(x - ct).$$

1. Suppose at $t = -\infty$ the particle is known to be in the ground state whose energy eigenfunction is $\langle x|\phi\rangle = u_i(x)$. Obtain the probability for finding the system in some excited state with energy eigenfunction $\langle x|\phi\rangle = u_f(x)$ at $t = +\infty$.
2. Interpret your result in (a) physically by regarding the δ-function pulse as a superposition of harmonic perturbations; recall

$$\delta(x - ct) = \frac{1}{2\pi c}\int_{-\infty}^{\infty} d\omega e^{i\omega(x/c - t)}.$$

   Emphasize the role played by energy conservation, which holds even quantum mechanically as long as the perturbation has been on for a very long time.

3.1. Solution

$c_f^{(1)}(t) = \frac{-i}{\hbar}\int_{t_0}^{t}\langle f|\,V_I(t')\,|i\rangle\,dt' = \frac{-i}{\hbar}\int_{t_0}^{t}\langle f|\,A\delta(x{-}ct')\,|i\rangle\,e^{i(E_f-E_i)t'/\hbar}dt' = \frac{-iA}{\hbar}\int_{t_0}^{t'}\langle f|\,\delta(x{-}ct)\,|i\rangle\,e^{i(E_f-E_i)t'/\hbar}dt'$

$= \frac{-iA}{\hbar}\int_{-\infty}^{\infty}\langle f|\,\delta(x-ct)\,|i\rangle\,e^{i(E_f-E_i)t'/\hbar}dt' = \frac{-iA}{\hbar}\int_{-\infty}^{\infty}\delta(x-ct')\,\langle f|i\rangle\,e^{i(E_f-E_i)t'/\hbar}dt'$

$= \frac{-iA}{\hbar}\int_{-\infty}^{\infty}\langle f|\,\delta(x-ct)\,|i\rangle\,e^{i(E_f-E_i)t'/\hbar}dt' = \frac{-iA}{\hbar}\int_{-\infty}^{\infty}dx\,|x\rangle\,\langle x|\int_{-\infty}^{\infty}dt'\delta(x-ct')\,\langle f|i\rangle\,e^{i(E_f-E_i)t'/\hbar}$

$= \frac{-iA}{\hbar}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}dxdt'\,\langle f|x\rangle\,\langle x|i\rangle\,\delta(x{-}ct')e^{i(E_f-E_i)t'/\hbar} = \frac{-iA}{\hbar}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}dxdt'\,\langle f|x\rangle\,\langle x|i\rangle\,\delta(x{-}ct')e^{i(E_f-E_i)t'/\hbar}$

$= \frac{-iA}{\hbar}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}dxdt'u_f^*(x)u_i(x)\delta(x-ct')e^{i(\omega_f-\omega_i)t'}$

Where the delta function will do its usual plucking out the function in the integral:

$= \frac{-iA}{\hbar}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}dxdt'u_f^*(x)u_i(x)\delta(x-ct')e^{i(\omega_f-\omega_i)t'} = \frac{-iA}{\hbar c}\int_{-\infty}^{\infty}dxu_f^*(x)u_i(x)e^{i(\omega_f-\omega_i)x/c}$

And the probability, as always:

$$|c_f^{(1)}(t)|^2 = \left|\frac{-iA}{\hbar c}\int_{-\infty}^{\infty}dxu_f^*(x)u_i(x)e^{i(\omega_f-\omega_i)x/c}\right|^2 = \boxed{\left(\frac{A}{\hbar c}\right)^2\left|\int_{-\infty}^{\infty}dxu_f^*(x)u_i(x)e^{i(\omega_f-\omega_i)x/c}\right|^2}$$

I think the implications here are clear as far as what the problem was aiming that we understood (and its really cool). The perturbation is a pulse that gives up energy $(E_f - E_i)/\hbar$, which is exactly the energy that the states get excited to. The energy difference given by the pulse, equals the energy gained by the particle. Energy is of course conserved!

4. 5.35

A hydrogen atom in its ground state $(n, l, m) = (1, 0, 0)$ is placed between the plates of a capacitor. A time-dependent but spatial uniform electric field (not potential!) is applied as follows:

$$E = \begin{cases} 0 & \text{for } t < 0 \\ E_0 e^{-t/\tau} & \text{for } t > 0 \quad (E_0 \text{ in the positive z-direction}). \end{cases}$$

Using first-order time-dependent perturbation theory, compute the probability for the atom to be found at $t \gg \tau$ in each of the three 2p states: $(n, l, m) = (2, 1, \pm1 \text{ or } 0)$. Repeat the problem for the 2s state: $(n, l, m) = (2, 0, 0)$. Consider the limit $t \to \infty$.

4.1. Solution:

We continue to study the hydrogen atom, and given that we are after the transition probability, we employ the result of time-dependant perturbation theory:
$c_n^{(0)}(t) = \delta_{ni}$
$c_n^{(1)}(t) = \frac{-i}{\hbar}\int_{t_0}^{t}\langle n|\,V_I(t')\,|i\rangle\,dt'$

But before we evaluate the previous integral, we need to find the potential energy of the field:

$U_E = q\Phi = -q\int \mathbf{E}\cdot d\mathbf{l} = -q\int E_0 e^{-t/\tau}dz = -qE_0 z e^{-t/\tau}$

We can now proceed for every state $|n\rangle = |2, 1, \pm1\rangle$ for the initial state $|i\rangle = |1, 0, 0\rangle$:

$c_{|21\pm1\rangle}^{(1)}(t) = \frac{qE_0 i}{\hbar}\int_{t_0}^{t}\langle 21\pm1|\,ze^{-t/\tau}(t')\,|100\rangle\,dt' = \frac{qE_0 i}{\hbar}\int_{t_0}^{t}\langle 21\pm1|\,ze^{-t/\tau}(t')\left[\frac{1}{\sqrt{\pi}}\left(\frac{1}{a_0}\right)^{\frac{3}{2}}e^{-r/a_0}\right]dt'$

$= \frac{qE_0 i}{\hbar}\int_{t_0}^{t}\left[\frac{1}{\sqrt{64\pi}}\left(\frac{1}{a_0}\right)^{3/2}(\frac{r}{a_0})e^{-r/2a_0}\sin(\theta)e^{\pm i\phi}\right] ze^{-t/\tau}(t')\left[\frac{1}{\sqrt{\pi}}\left(\frac{1}{a_0}\right)^{\frac{3}{2}}e^{-r/a_0}\right]dt' = \boxed{0}$ by Wigner-Eckart

$c^{(1)}_{|200\rangle}(t) = \frac{qE_0 i}{\hbar}\int_{t_0}^{t}\langle 200|\, ze^{-t/\tau}(t')\,|100\rangle\, dt' = \boxed{0}$ by parity

In summary, no transition is allowed for $|21\pm 1\rangle$ or $|200\rangle$

$c^{(1)}_{|210\rangle}(t) = \frac{qE_0 i}{\hbar}\int_{t_0}^{t}\langle 210|\, ze^{-t/\tau}(t')\,|100\rangle\, dt' = \frac{qE_0 i}{\hbar}\int_{t_0}^{t}\left[\frac{1}{\sqrt{32\pi}}\left(\frac{1}{a_0}\right)^{\frac{3}{2}}\frac{r}{a_0}e^{-\frac{r}{2a_0}}\cos(\theta)\right] ze^{-t/\tau}(t')\left[\frac{1}{\sqrt{\pi}}\left(\frac{1}{a_0}\right)^{\frac{3}{2}}e^{-r/a_0}\right]dt'$

$= \frac{qE_0 i}{\hbar}\frac{128\sqrt{2}}{243}a_0\int_0^t e^{(i\omega-1/\tau)t'} = \frac{qE_0 i}{\hbar}\frac{128\sqrt{2}}{243}a_0\left[\frac{1}{i\omega-1/\tau}\left(e^{(i\omega-1/\tau)t}-1\right)\right]$

$|c^{(1)}_{|210\rangle}(t)|^2 = \left|\frac{qE_0 i}{\hbar}\frac{128\sqrt{2}}{243}a_0\left[\frac{1}{i\omega-1/\tau}\left(e^{(i\omega-1/\tau)t}-1\right)\right]\right|^2 = \left(\frac{qE_0}{\hbar}\frac{128\sqrt{2}}{243}a_0\right)^2\left|\left[\frac{1}{1/\tau-\omega}\left(1-e^{(\omega-1/\tau)t}\right)\right]\right|^2$

$$= \boxed{\left(\frac{qE_0}{\hbar}\frac{128\sqrt{2}}{243}a_0\right)^2\left|\left[\frac{1}{1/\tau-\omega}\left(1-e^{(\omega-1/\tau)t}\right)\right]\right|^2}$$

And as we take $t\to\infty$

$$|c^{(1)}_{|210\rangle}(t\to\infty)|^2 = \left(\frac{qE_0}{\hbar}\frac{128\sqrt{2}}{243}a_0\right)^2\left|\left[\frac{1}{1/\tau-\omega}(1)\right]\right|^2 = \boxed{\left(\frac{qE_0}{\hbar}\frac{128\sqrt{2}}{243}a_0\right)^2\left|\left[\frac{1}{1/\tau-\omega}\right]\right|^2}$$

5. 5.36

Consider a composite system made up of two spin $\frac{1}{2}$ objects. For $t<0$, the Hamiltonian does not depend on spin and can be taken to be zero by suitably adjusting the energy scale. For $t>0$, the Hamiltonian is given by

$$H = \left(\frac{4\Delta}{\hbar^2}\right)\mathbf{S}_1\cdot\mathbf{S}_2.$$

Suppose the system is in $|+-\rangle$ for $t\le 0$. Find, as a function of time, the probability for being found in each of the following states $|++\rangle$, $|+-\rangle$, $|-+\rangle$, and $|--\rangle$.

1. By solving the problem exactly.
2. By solving the problem assuming the validity of first-order time-dependent perturbation theory with H as a perturbation switched on at $t=0$. Under what condition does (b) give the correct results?

5.1. Solution

To solve this exactly, we will need to re-write the Hamiltonian as a cross term:

$\hat{H} = \frac{4\Delta}{\hbar^2}\hat{S}_1\cdot\hat{S}_2 = \frac{2\Delta}{\hbar^2}(\hat{S}^2-\hat{S}_1^2-\hat{S}_2^2)$

Then evolve the state through the unitary time evolution operator with knowing how the spin-squared operators hit the eigenstates $\mathbf{S^2}|s,m\rangle = \hbar^2 S(S+1)|s,m\rangle$. Knowing the initial state, I also re-write it as a superposition of eigenstates:

$|\alpha(0)\rangle = \frac{1}{\sqrt{2}}\left[|1,0\rangle+|0,0\rangle\right]$

$|\alpha(t)\rangle = e^{-iHt/\hbar}|\alpha(0)\rangle = \frac{1}{\sqrt{2}}e^{-iHt/\hbar}\left[|1,0\rangle+|0,0\rangle\right] = \frac{1}{\sqrt{2}}e^{-2i\Delta(S^2-S_1^2-S_2^2)t/\hbar^3}\left[|1,0\rangle+|0,0\rangle\right] =$

$= \frac{1}{\sqrt{2}}\left[e^{-2i\Delta(\frac{2}{4}\hbar^2)t/\hbar^3}|1,0\rangle + e^{-2i\Delta(\frac{6}{4}\hbar^2)t/\hbar^3}|0,0\rangle\right] = \boxed{\frac{1}{\sqrt{2}}\left[e^{-i\Delta t/\hbar}|1,0\rangle + e^{3i\Delta t/\hbar}|0,0\rangle\right]}$

From seeing the superposition states of the time evolved states, its obvious that two of the probabilities we are after vanish due to orthogonality of the states. Namely,

$$\mathbf{P}(|{+}{+}\rangle) = |\langle{+}{+}|{+}{-}\rangle|^2 = \boxed{0}$$

$$\mathbf{P}(|{-}{-}\rangle) = |\langle{-}{-}|{+}{-}\rangle|^2 = \boxed{0}$$

$$\mathbf{P}(|{+}{-}\rangle) = |\langle{+}{-}|{+}{-}\rangle|^2 = \tfrac{1}{4}\left|[\langle 00| + \langle 10|]\left[e^{-i\Delta t/\hbar}|10\rangle + e^{3i\Delta t/\hbar}|00\rangle\right]\right|^2 = \tfrac{1}{4}\left|e^{-i\Delta t/\hbar} + e^{3i\Delta t/\hbar}\right|^2 = \boxed{\frac{1}{2}\left(1 + \cos(\frac{4\Delta t}{\hbar})\right)}$$

$$\mathbf{P}(|{-}{+}\rangle) = |\langle{-}{+}|{+}{-}\rangle|^2 = \tfrac{1}{4}\left|[\langle 10| - \langle 00|]\left[e^{-i\Delta t/\hbar}|10\rangle + e^{3i\Delta t/\hbar}|00\rangle\right]\right|^2 = \tfrac{1}{4}\left|e^{-i\Delta t/\hbar} - e^{3i\Delta t/\hbar}\right|^2 = \boxed{\frac{1}{2}\left(1 - \cos(\frac{4\Delta t}{\hbar})\right)}$$

Using perturbation theory with treating H as the perturbation:

$$c^{(1)}_{++}(t) = \tfrac{-i}{\hbar}\int_0^t \langle{+}{+}|\, e^{iHt/\hbar}V(t')e^{-iHt/\hbar}|{+}{-}\rangle\, dt' = \tfrac{-i}{\hbar}\int_0^t \langle{+}{+}|\, e^{iHt/\hbar}\left(\tfrac{2\Delta}{\hbar^2}(\hat{S}^2 - \hat{S}_1^2 - \hat{S}_2^2)\right)e^{-iHt/\hbar}|{+}{-}\rangle\, dt'$$
$$= \tfrac{-i}{\hbar}\tfrac{2\Delta}{\hbar^2}\int_0^t \langle{+}{+}|\, e^{iHt/\hbar}(\hat{S}^2 - \hat{S}_1^2 - \hat{S}_2^2)e^{-iHt/\hbar}|{+}{-}\rangle\, dt' = \boxed{0}$$

Due to orthonormality of the states. Likewise,

$$c^{(1)}_{--}(t) = \tfrac{-i}{\hbar}\tfrac{2\Delta}{\hbar^2}\int_0^t \langle{-}{-}|\, e^{iHt/\hbar}(\hat{S}^2 - \hat{S}_1^2 - \hat{S}_2^2)e^{-iHt/\hbar}|{+}{-}\rangle\, dt' = \boxed{0}$$

$$c^{(1)}_{-+}(t) = \tfrac{-i}{\hbar}\tfrac{2\Delta}{\hbar^2}\int_0^t \langle{-}{+}|\, e^{iHt/\hbar}(\hat{S}^2 - \hat{S}_1^2 - \hat{S}_2^2)e^{-iHt/\hbar}|{+}{-}\rangle\, dt' = \tfrac{-2i\Delta}{\hbar^3}\int_0^t \langle{-}{+}|\,(\hat{S}^2 - \hat{S}_1^2 - \hat{S}_2^2)\,|{+}{-}\rangle\, dt'$$
$$= \tfrac{-2i\Delta}{\hbar^3}\int_0^t \langle{-}{+}|\,(\hat{S}^2 - \hat{S}_1^2 - \hat{S}_2^2)\,|{+}{-}\rangle\, dt' = \tfrac{-2i\Delta}{\hbar^3}\int_0^t \left[\tfrac{1}{\sqrt{2}}\langle 10| - \langle 00|\right](\hat{S}^2 - \hat{S}_1^2 - \hat{S}_2^2)\left[\tfrac{1}{\sqrt{2}}|10\rangle + |00\rangle\right] dt'$$
$$= \tfrac{-i\Delta}{\hbar^3}\int_0^t [\langle 10| - \langle 00|]\,(\hat{S}^2 - \hat{S}_1^2 - \hat{S}_2^2)\,[|10\rangle + |00\rangle]\, dt' = \tfrac{-i\Delta}{\hbar^3}\left[\tfrac{1}{2}t\hbar^2 + \tfrac{3}{2}t\hbar^2\right] = \boxed{\frac{-2i\Delta}{\hbar}t}$$

Hence,

$$|c^{(1)}_{-+}(t)|^2 = |\tfrac{-2i\Delta}{\hbar}t|^2 = \boxed{\frac{4\Delta^2 t^2}{\hbar^2}}$$

and,

$$c^{(1)}_{+-}(t) = \tfrac{-i\Delta}{\hbar^3}\int_0^t [\langle 10| + \langle 00|]\,(\hat{S}^2 - \hat{S}_1^2 - \hat{S}_2^2)\,[|10\rangle + |00\rangle]\, dt' = \tfrac{-i\Delta}{\hbar^3}\left[\tfrac{1}{2}t\hbar^2 - \tfrac{3}{2}t\hbar^2\right] = \boxed{\frac{2i\Delta}{\hbar}t}$$

$$|c^{(1)}_{+-}(t)|^2 = |\tfrac{2i\Delta}{\hbar}t|^2 = \boxed{\frac{-4\Delta^2 t^2}{\hbar^2}}$$

This gives the correct results when the time passed it sufficiently small enough. For large t, the time-dependant approximation's accuracy diminishes.

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HOMEWORK 5 — Hamiltonians with Extreme Time Dependence and Berry's Phase

1. 5.28

Show that the slow-turn-on of perturbation $V \to Ve^{\eta t}$ (see Baym (1969), p. 257) can generate contribution from the second term in (5.295).

1.1. Solution

Plug and chug:

$$c_n^{(2)}(t) = \left(\frac{-i}{\hbar}\right)^2 \sum_m \int_{-\infty}^{t} dt' \int_{-\infty}^{t'} dt'' e^{i\omega_{nm}t'} \langle n|V(t')|m\rangle (t') e^{i\omega_{mi}t''} \langle m|V(t'')|i\rangle$$
$$= \left(\frac{-i}{\hbar}\right)^2 \sum_m \int_{-\infty}^{t} dt' \int_{-\infty}^{t'} dt'' e^{i\omega_{nm}t'} \left\langle n|Ve^{\eta t'}|m\right\rangle (t') e^{i\omega_{mi}t''} \left\langle m|Ve^{\eta t''}|i\right\rangle$$
$$= \left(\frac{-i}{\hbar}\right)^2 \sum_m \int_{-\infty}^{t} dt' \int_{-\infty}^{t'} dt'' e^{i\omega_{nm}t'} e^{\eta t} \langle n|V|m\rangle e^{i\omega_{mi}t''} e^{\eta t} \langle m|V|i\rangle$$
$$= \left(\frac{-i}{\hbar}\right)^2 \sum_m \int_{-\infty}^{t} dt' \int_{-\infty}^{t'} dt'' e^{i\omega_{nm}t'} e^{\eta t'} V_{nm} e^{i\omega_{mi}t''} e^{\eta t''} V_{mi}$$
$$= \left(\frac{-i}{\hbar}\right)^2 \sum_m \int_{-\infty}^{t} e^{(i\omega_{nm}+\eta)t'} dt' \int_{-\infty}^{t'} dt'' V_{nm} e^{(i\omega_{mi}+\eta)t''} V_{mi}$$
$$= \left(\frac{-i}{\hbar}\right)^2 \sum_m V_{nm}V_{mi} \left[\frac{1}{i\omega_{nm}+\eta}\left(e^{(i\omega_{nm}+\eta)t} - e^{-\infty}\right)\right] \int_{-\infty}^{t'} dt'' e^{(i\omega_{mi}+\eta)t''}$$
$$= \left(\frac{-i}{\hbar}\right)^2 \sum_m V_{nm}V_{mi} \left[\frac{1}{i\omega_{nm}+\eta}\left(e^{(i\omega_{nm}+\eta)t} - \cancel{e^{-\infty}}\right)\right] \left[\frac{1}{i\omega_{mi}+\eta}\left(e^{(i\omega_{mi}+\eta)t'} - \cancel{e^{-\infty}}\right)\right]$$
$$= \left(\frac{-i}{\hbar}\right)^2 \sum_m V_{nm}V_{mi} \left[\frac{1}{i\omega_{nm}+\eta}\frac{1}{i\omega_{mi}+\eta} e^{(i\omega_{nm}+\eta)t} e^{(i\omega_{mi}+\eta)t'}\right]$$
$$= \left(\frac{i}{\hbar}\right)^2 \frac{e^{(i\omega_{ni}+2\eta)t}}{i\omega_{ni}+2\eta} \sum_m V_{nm}V_{mi} \frac{1}{i\omega_{mi}+\eta}$$

Using $\omega = \frac{E}{\hbar}$

$$= \left(\frac{i}{\hbar}\right)^2 \frac{e^{(i\omega_{ni}+2\eta)t}}{i\frac{E_{ni}}{\hbar}+2\eta} \sum_m V_{nm}V_{mi} \frac{1}{i\frac{E_{mi}}{\hbar}+\eta} = e^{i\omega_{ni}t} e^{2\eta t} \frac{1}{E_n - E_i - 2i\hbar\eta} \sum_m \frac{V_{nm}V_{mi}}{E_m - E_i - i\hbar\eta}$$
$$= \frac{e^{i\omega_{ni}t}e^{2\eta t} - 1}{E_n - E_i - 2i\hbar\eta} \sum_m \frac{V_{nm}V_{mi}}{E_m - E_i - i\hbar\eta} + \frac{1}{E_n - E_i - 2i\hbar\eta} \sum_m \frac{V_{nm}V_{mi}}{E_m - E_i - i\hbar\eta}$$
$$= \frac{1}{E_n - E_i - 2i\hbar\eta} \sum_m \frac{V_{nm}V_{mi}}{E_m - E_i - i\hbar\eta} + \frac{e^{i\omega_{ni}t}e^{2\eta t} - 1}{E_n - E_i - 2i\hbar\eta} \sum_m \frac{V_{nm}V_{mi}}{E_m - E_i - i\hbar\eta}$$

Which is just re-writing the expression to explicitly show that when $\eta \to 0$, we get contribution from the second term in the form of:

$$\frac{e^{i\omega_{ni}t}\cancel{e^{2\eta t}} - 1}{E_n - E_i - \cancel{2i\hbar\eta}} \sum_m \frac{V_{nm}V_{mi}}{E_m - E_i - \cancel{i\hbar\eta}} = \frac{e^{i\omega_{ni}t} - 1}{E_n - E_i} \sum_m \frac{V_{nm}V_{mi}}{E_m - E_i}$$

With the factor in front of the sum as:

$$\boxed{\frac{e^{i\omega_{ni}t} - 1}{E_n - E_i} \text{ when } \eta \to 0}$$

2. 5.42

Consider an atom made up of an electron and a singly charged ($Z = 1$) triton (^{3}H). Initially the system is in its ground state ($n = 1, l = 0$). Suppose the system undergoes beta decay, in which the nuclear charge *suddenly* increases by one unit (realistically by emitting an electron and an antineutrino). This means that the tritium nucleus (called a "triton") turns into a helium ($Z = 2$) nucleus of mass 3 (^{3}He).

- Obtain the probability for the system to be found in the ground state of the resulting helium ion.
- The available energy in tritium beta decay is about 18 keV and the size of the ^{3}He atom is about 1 Å. Check that the time scale T for the transformation satisfies the criterion of validity for the sudden approximation.

2.1. Solution

2.1.1. part(a.)

We can approximate this as a sudden approximation, thus only leaving us with the probability to calculate as the inner product of the initial and final states:
Using hydrogen-like wavefunctions for the ground states,

$$\psi_{100} = \frac{1}{\sqrt{\pi}}\left(\frac{Z}{a_0}\right)^{3/2} e^{-Zr/a_0}$$

with a_0 as the usual Bohr radius and Z as the atomic number.

$$\mathbf{P_{m\to n}} = \left|\left\langle \phi_n^{'}|\phi_m\right\rangle\right|^2 = \left|\int \left[\frac{1}{\sqrt{\pi}}\left(\frac{2}{a_0}\right)^{3/2} e^{-2r/a_0}\right]\left[\frac{1}{\sqrt{\pi}}\left(\frac{1}{a_0}\right)^{3/2} e^{-r/a_0}\right] r^2 \sin\theta dr d\theta d\phi\right|^2 = \left|\frac{1}{\pi}\frac{2^{3/2}}{a_0^3}\int e^{-3r/a_0} r^2 \sin\theta dr d\theta d\phi\right|^2$$
$$= \left|\frac{1}{\pi}\frac{2^{3/2}}{a_0^3}\int_0^\pi \sin\theta d\theta \int_0^{2\pi} d\phi \int_0^\infty e^{-3r/a_0} r^2 dr\right|^2 = \left|\frac{1}{\pi}\frac{2^{3/2}}{a_0^3} 4\pi \int_0^\infty e^{-3r/a_0} r^2 dr\right|^2 = \left|\frac{1}{\pi}\frac{2^{3/2}}{a_0^3} 4\pi \frac{2a_0^3}{27}\right|^2 = \left|\frac{8\cdot 2^{3/2}}{27}\right|^2$$
$$\approx \boxed{0.702331961591}$$

2.1.2. part(b.)

Here we just have to show that $T = \frac{d}{v} = \frac{1A}{v} << \frac{2\pi}{\omega}$

Where,

$$v = \sqrt{\frac{2E}{m}} = \sqrt{\frac{2(18\cdot 10^3 eV)(1.602\cdot 10^{-19} J/eV)}{9.1\cdot 10^{-31} kg}} \approx 7.95\cdot 10^7 m/s$$

$$T = \frac{1\cdot 10^{-10} m}{7.95\cdot 10^7 m/s} \approx \boxed{1.256\cdot 10^{-18} s}$$

and,

$$\Delta\omega = \Delta E/\hbar = \frac{(54.4eV - 13.6eV)\left(1.602\cdot 10^{-19} J/eV\right)}{1.0545\cdot 10^{-34} Js} \approx 6.198\cdot 10^{16}\text{Hz}$$

$$2\pi/\Delta\omega = 2\pi/6.198\cdot 10^{16}\text{ Hz} = \boxed{1.0136867681\cdot 10^{-16} s}$$

Where then it becomes clear that,

$$\boxed{T = 1.256\cdot 10^{-18} s << \frac{2\pi}{\omega} = 1.0136867681\cdot 10^{-16} s}$$

3. 5.43

Show that $\mathbf{A_n}(\mathbf{R})$ defined in (5.234) is a purely real quantity.

3.1. Solution

This problem is effectively done in the text:

$$\mathbf{A}_n(\mathbf{R}) = i\langle n,t|[\nabla_{\mathbf{R}}|n,t\rangle]$$

$$\nabla_{\mathbf{R}} i\langle nt|nt\rangle = i[\nabla_{\mathbf{R}}\langle nt]|nt\rangle + i\langle nt[\nabla_{\mathbf{R}}|nt\rangle] = 0$$

$$i\langle nt[\nabla_{\mathbf{R}}|nt\rangle] = -i[\nabla_{\mathbf{R}}\langle nt]|nt\rangle = (i\langle nt[\nabla_{\mathbf{R}}|nt\rangle])^*$$

$$(\mathbf{A}_n(\mathbf{R}))^* = (i\langle n,t|[\nabla_{\mathbf{R}}|n,t\rangle])^* = -i[\langle n,t|\nabla_{\mathbf{R}}]|n,t\rangle = i\langle n,t|[\nabla_{\mathbf{R}}|n,t\rangle] = \boxed{\mathbf{A}_n(\mathbf{R})}$$

4. 5.44

Consider a neutron in a magnetic field, fixed at an angle θ with respect to the z-axis, but rotating slowly in the ϕ direction. That is, the tip of the magnetic field traces out a circle on the surface of the sphere, at "latitude"$\pi - \theta$. Explicitly calculate the Berry potential A for the spin-up state from (5.234), take its curl, and determine Berry's phase γ_+. Thus, verify (5.253) for this particular example of a curve C. (For hints, see "The adiabatic theorem and Berry's phase"by Holstein, *Am. J. Phys.*, 57 (1989) 1079.)

4.1. Solution

Knowing,

$$\mathbf{A}_n(\mathbf{R}) = i\langle n,t|[\nabla_{\mathbf{R}}|n,t\rangle] = i\begin{bmatrix}\cos\theta/2 & e^{-i\phi}\sin\theta/2\end{bmatrix}\left(\cancel{\frac{\partial}{\partial r}\hat{r}} + \frac{1}{r}\frac{\partial}{\partial\theta}\hat{\theta} + \frac{1}{r\sin\theta}\frac{\partial}{\partial\phi}\hat{\phi}\right)\begin{bmatrix}\cos\theta/2 \\ e^{i\phi}\sin\theta/2\end{bmatrix}$$

$$= i\begin{bmatrix}\cos\theta/2 & e^{-i\phi}\sin\theta/2\end{bmatrix}\left(\frac{1}{2r}\begin{bmatrix}-\sin\theta/2 \\ e^{i\phi}\cos\theta/2\end{bmatrix}\hat{\theta} + \frac{1}{r\sin\theta}\begin{bmatrix}0 \\ ie^{i\phi}\sin\theta/2\end{bmatrix}\hat{\phi}\right) = \frac{i}{2r}\left[-\cos\theta/2\sin\theta/2 + \cos\theta/2\sin\theta/2\right]\hat{\theta} +$$

$$\frac{i}{r\sin\theta}\left[i\sin^2\theta/2\right]\hat{\phi} = \frac{-1}{r\sin\theta}\sin^2(\theta/2)\hat{\phi} = \boxed{\frac{\cos\theta - 1}{2r\sin\theta}\hat{\phi}}$$

$$\mathbf{B}(\mathbf{r}) = \nabla\times\mathbf{A}(\mathbf{r}) = \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(A_\phi\sin\theta)\hat{r} - \frac{1}{r}\frac{\partial}{\partial r}(rA_\phi)\hat{\theta} = \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}\left(\frac{\cos\theta-1}{2r\sin\theta}\sin\theta\right)\hat{r} - \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\cos\theta-1}{2r\sin\theta}\right)\hat{\theta}$$

$$= \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}\left(\frac{\cos\theta-1}{2r}\right)\hat{r} - \frac{1}{r}\frac{\partial}{\partial r}\left(\frac{\cos\theta-1}{2\sin\theta}\right)\hat{\theta} = \boxed{-\frac{1}{2r^2}\hat{r}}$$

$\gamma_+ = \int\nabla\times\mathbf{A}(\mathbf{R})\cdot d\mathbf{a} = -\frac{1}{2}\int\frac{\hat{r}}{r^2}\cdot d\mathbf{a} = \boxed{-\frac{1}{2}\Omega}$ Which is in complete agreement with 5.253,and Ω is the solid angle swept out by the cone made by the B-field.

5. 5.49

Calculate the lifetimes for the $2p \to 1s$ and $3p \to 1s$ transitions in the hydrogen atom. You can find measurements of these lifetimes in Bickel and Goodman, *Phys. Rev.*, 148 (1966) 1.

5.1. Solution

The following is done by knowing that these transitions are mediated through the photon on behalf of energy in the electromagnetic fields. As such, the perturbation which causes the transition is the Hamiltonian given by the interaction between the EM field and the atom. I approximate the interaction as an electric dipole in an electric field, to represent roughly the electron cloud in the electric field component of the light.

$$w_{i\to n} = \left(\frac{2\pi}{\hbar}\right)|\bar{V_{ni}}|^2\rho(E_n)_{E_n\approx E_i} = \left(\frac{2\pi}{\hbar}\right)|\langle n|V|i\rangle|^2\rho(E_n)_{E_n\approx E_i} = \left(\frac{2\pi}{\hbar}\right)|\langle n|V|i\rangle|^2\frac{\omega^2}{2\pi^2\hbar c^3}L^3dE = \frac{\omega^2}{\pi\hbar^2c^3}L^3|\langle n|V|i\rangle|^2dE$$

And then invoke the result of the work of Sakurai and Napolitano in eq 5.359:

$\frac{1}{\tau_{p \longrightarrow s}} = \frac{4}{9}\alpha\frac{\omega^3}{c^2}|R_{\alpha',\alpha}|^2$ where

$R_{\alpha',\alpha} = \int_0^\infty r^2 dr R^*_{\alpha'}(r) r R_\alpha(r) = \int_0^\infty r^3 dr R^*_{\alpha'}(r) R_\alpha(r) = \frac{2}{\sqrt{3}a_0}\left(\frac{1}{a_0}\right)^{3/2}\left(\frac{1}{2a_0}\right)^{3/2}\int_0^\infty e^{-3r/2a_0} r^4 dr$

$\approx 6.83 \cdot 10^{-11} m$

$\frac{1}{\tau_{p \longrightarrow s}} = \frac{4}{9}\alpha\frac{\omega^3}{c^2}|R_{\alpha',\alpha}|^2 = \frac{4}{9}\left(\frac{1}{137}\right)\frac{\Delta E}{c^2}\left(6.83 \cdot 10^{-11} m\right)^2 = \frac{4}{9}\left(\frac{1}{137}\right)\frac{-3 \cdot 13.6\left(1.6 \cdot 10^{-19} J\right)}{4(3 \cdot 10^8 m/s)^2}\left(6.83 \cdot 10^{-11} m\right)^2 \approx 630546265.539 s^{-1}$

$\tau_{p \longrightarrow s} = \frac{1}{630546265.539 s^{-1}} \approx \boxed{1.6 \cdot 10^{-9} s}$

And for the $3p$:

$R_{\alpha',\alpha} = \int_0^\infty r^3 dr R^*_{\alpha'}(r) R_\alpha(r) = \int_0^\infty r^3 \left[\frac{4}{81\sqrt{6}a_0^{3/2}}\left(6 - \frac{r}{a_0}\right)\frac{r}{a_0}e^{-r/3a_0}\right]\left[\frac{2}{a_0^{3/2}}e^{-r/a_0}\right] dr = \frac{81}{64\sqrt{6}}a_0$

$\frac{1}{\tau} = \frac{4}{9}\alpha\frac{\omega^3}{c^2}|R|^2 = \frac{4}{9}\frac{1}{137}\frac{(\Delta E)^3}{c^2}\left|\frac{81}{64\sqrt{6}}a_0\right|^2 = \frac{4}{9}\frac{1}{137}\frac{\left(\frac{-8}{9}E_0\hbar\right)^3}{(3 \cdot 10^8 m/s)^2}\left|\frac{81}{64\sqrt{6}}a_0\right|^2 \approx 168335532.56 s^{-1}$

$\tau = (168335532.56)^{-1} s \approx \boxed{5.9 \cdot 10^{-9} s}$

Submitted by Delano Campos on .